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Recursive Kinematics and Inverse Dynamics for a Planar 3R Parallel Manipulator

We focus on the development of modular and recursive formulations for the inverse dynamics of parallel architecture manipulators in this paper. The modular formulation of mathematical models is attractive especially when existing sub-models may be assembled to create different topologies, e.g., cooperative robotic systems. Recursive algorithms are desirable from the viewpoint of simplicity and uniformity of computation. However, the prominent features of parallel architecture manipulators-the multiple closed kinematic loops, varying locations of actuation together with mixtures of active and passive joints-have traditionally hindered the formulation of modular and recursive algorithms. In this paper, the concept of the decoupled natural orthogonal complement (DeNOC) is combined with the spatial parallelism of the robots of interest to develop an inverse dynamics algorithm which is both recursive and modular. The various formulation stages in this process are highlighted using the illustrative example of a 3R Planar Parallel Manipulator. [DOI: 10.1115/1.2098890]

Introduction

The *modular* and *recursive* inverse dynamics formulation for parallel architecture manipulators is the subject of this paper. The *modular* formulation of mathematical models is attractive because existing submodels may be assembled to create different topologies, e.g., cooperative robotic systems. *Recursive* algorithms are desirable from the viewpoint of simplicity and uniformity of computation, despite the ever-increasing complexity of mechanisms. We also note that, prior to the dynamics computation stage, a forward or inverse kinematics stage is often required. Hence, the development of efficient recursive dynamics algorithms also necessitates the careful investigation of *recursive kinematics algorithms*. However, the development of *modular* and *recursive* kinematics and dynamics algorithms for parallel manipulator architectures remains a challenging research problem.

The literature on mathematical modeling of manipulators has a rich history spanning several decades. We will summarize some critical aspects presently while referring the interested reader to any number of books on the subject [1–5], for details. Methods for formulation of equations of motion (EOM) fall into two main categories: (a) Euler-Lagrange and (b) Newton-Euler formulations. *Euler-Lagrange* methods are commonly used in the robotics community to obtain the equations of motion (EOM) of robotic manipulators. Typically, such approaches use the joint-based relative coordinates as the configuration space. For serial chain manipulators, these form a minimal coordinate description and permit a direct mapping to actuator coordinates. *Newton-Euler* (NE) methods, on the other hand, typically favor the use of Cartesian variables as configuration-space variables, and develop recursive formulations from the free-body diagram of each single body. The *uncoupled* governing equations are then assembled to obtain the model of the entire system.

While efficient formulations exist for serial-chain and tree-structured multibody systems, the adaptation of these methods to the simulation of closed-chain linkages and parallel manipulators is relatively more difficult. Such systems possess one or more

kinematic loops, requiring the introduction of algebraic constraints, typically nonlinear, into the formulation. Considerable work has been reported in the literature on the specialization of the above methods to formulate the EOM of constrained mechanical systems, while including both holonomic and nonholonomic constraints. Parallel mechanisms and manipulators form a special class of constrained mechanical systems where the multiple kinematic loops give rise to systems of nonlinear holonomic constraints; in the ensuing discussion we will focus on the development of EOM of such systems.

Nonrecursive Newton-Lagrange Formulations. The dynamics of constrained mechanical systems with closed loops using the Newton-Lagrange approach is traditionally obtained by cutting the closed loops to obtain various open loops and tree-structured systems, and then writing a system of ordinary differential equations (ODEs) for the corresponding chains in their corresponding generalized coordinates [6]. The solution to these equations are required to satisfy additional algebraic equations guaranteeing the closure of the cut-open loops. A Lagrange multiplier term is introduced to represent the forces in the direction of the constraint violation. The resulting formulation, often referred to as a *descriptor form*, yields a usually simpler, albeit larger, system of index-3 differential algebraic equations (DAEs).

Typical methods used to solve the forward and inverse dynamics problems for such constrained systems cover a broad spectrum, namely,

- Direct elimination where the surplus variables are eliminated directly, using the position-level algebraic constraints to explicitly reduce index-3 DAE to an ODE in a minimal set of generalized coordinates (conversion into Lagrange's equations of the second kind) [7];
- Explicit Lagrange-multiplier computation together with the unknown accelerations computed from the augmented index-1 differential algebraic equation (DAE) formed by appending the differentiated acceleration level constraints to the system equations [1,8];
- Lagrange-multiplier approximation/penalty formulation, where the Lagrange multipliers are estimated using a compliance-based force law, based on the extent of constraint violation and assumed penalty factors [2,9];
- Projection of dynamics onto the tangent space of the constraint manifold, where the constraint-reaction dynamics

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equations are taken into the orthogonal and tangent subspaces of the vector space of the system generalized velocities. A family of choices exist for the projection, as surveyed by García de Jalón and Bayo [2] and Shabana [5].

Recursive Newton-Euler Formulations. Many variants of fast and readily implementable recursive algorithms have been formulated within the last two decades, principally within the robotics community to overcome the limitations posed by the complexity of the dynamics equations based on classic Lagrange approaches [6].

The first researchers to develop O(N) algorithms for inverse dynamics for robotics used a Newton-Euler formulation of the problem. Stepanenko and Vukobratovic [10] developed a recursive NE method for human limb dynamics, while Orin et al. [11] improved the efficiency of the recursive method by referring forces and moments to local link coordinates for the real-time control of a leg of a walking machine. Luh et al. [12] developed a very efficient recursive Newton-Euler algorithm by referencing most quantities to link coordinates; this is the most often cited. Further gains have been made in the efficiency over the years, as reported, for example, by Balafoutis et al. [13] and Goldenberg and He [14].

In multiloop mechanisms, the generalized coordinates (joint variables) are no longer independent, since they are subject to the typically nonlinear loop-closure constraints. The most common method for dealing with kinematics is to cut the loop, introduce Lagrange multipliers to substitute for the cut joints, and use a recursive scheme for the open-chain system to obtain a recursive algorithm.

Decoupled Natural Orthogonal Complement. The *natural orthogonal complement* (NOC), introduced in Angeles and Lee [15], belongs to the class of projection methods for dynamics evaluation. Saha [16] showed a method for splitting the NOC of a serial chain into two matrices, one diagonal and one lower block diagonal, thus introducing the *decoupled natural orthogonal complement* (DeNOC). By doing so, Saha was able to exploit the recursive nature of the DeNOC and apply the concept to model simple serial manipulators. Further, although recursive kinematics algorithms for serial chains have had a long history [10–12], a recursive algorithm for the forward kinematics of *closed-chain systems* appeared in Saha and Schiehlen [17]. In this work, Saha and Schiehlen [17] showed that the NOC of a parallel manipulator may be split into three parts—one full, one block diagonal, and one lower triangular, and proposed a recursive minimal-order forward dynamics algorithm for parallel manipulators. Examples of up to two degrees-of-freedom planar manipulators were included and various physical interpretations were reported.

Background

Twists, Wrenches, and Equations of Motion. In this section, some definitions and concepts associated with the formulation of the kinematics and dynamics of articulated rigid body systems coupled by lower kinematic pairs will be briefly reviewed. See [18,19] for further details.

Figure 1 shows two rigid links connected by a kinematic pair. The mass center of the ith link is at C_i while that of link $i-1$ is at C_{i-1}. The axis of the ith pair is represented by a unit vector $\mathbf{e}_i$. We attach a frame $\mathcal{F}_i$ with origin O_i and axes $\mathbf{x}_i$, $\mathbf{y}_i$, and $\mathbf{z}_i$, to link $(i-1)$ such that $\mathbf{z}_i$ is along $\mathbf{e}_i$. The global inertial reference frame $\mathcal{F}_0$ with axes $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ is attached to the base of the manipulator, and unless otherwise specified, all quantities will be represented in this global frame in the balance of the paper. Further, we define, the three-dimensional position vectors $\mathbf{d}_i$ from the O_i to the mass center of link i and $\mathbf{r}_{i-1}$ from the mass center of link $i-1$ to O_i.

The six-dimensional twist and wrench vector associated with link i, at its mass center C_i, are now defined
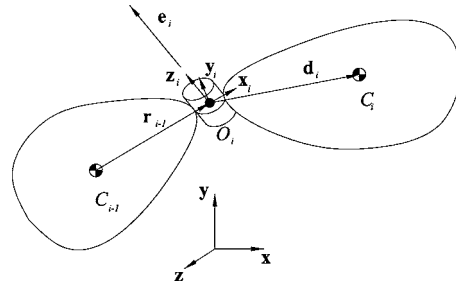

Fig. 1 Two bodies connected by a kinematic pair

$$\mathbf{t}_i = \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix}; \quad \mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{1}$$

where $\boldsymbol{\omega}$, $\mathbf{v}$, $\mathbf{n}$, and $\mathbf{f}$ are three-dimensional angular velocity, linear velocity, moment, and force vectors, respectively, associated with link i and represented about C_i. The Newton-Euler equations for link i are

$$\mathbf{n}_i = \mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \dot{\mathbf{I}}_i \boldsymbol{\omega}_i = \mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i$$

$$\mathbf{f}_i = m_i \dot{\mathbf{v}}_i$$

where $\mathbf{I}_i$ is the 3×3 inertia tensor about C_i and m_i is the mass of link i. The above set, in matrix form, may be written as

$$\mathbf{w}_i = \underbrace{\begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix}}_{\mathbf{M}_i} \begin{bmatrix} \dot{\boldsymbol{\omega}}_i \\ \dot{\mathbf{v}}_i \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}}_{\mathbf{W}_i} \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \tag{2}$$

or

$$\mathbf{w}_i = \mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{t}_i \tag{3}$$

For a multibody system with n rigid links coupled by kinematic pairs, we may write

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_n \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix} \tag{4}$$

The resulting set of Newton-Euler equations for the entire unconstrained system is then cast in the form

$$\mathbf{w} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{WMt} \tag{5}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M}_n \end{bmatrix}; \quad \mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{W}_n \end{bmatrix} \tag{6}$$

Kinematic Relations Between Two Bodies Coupled by Lower Kinematic Pair. The twist of link i at O_i can be written recursively in terms of the twist of link $i-1$ at C_{i-1} as

$$\tilde{\mathbf{t}}_i = \tilde{\mathbf{B}}_{i-1}\mathbf{t}_{i-1} + \tilde{\mathbf{p}}_i \dot{\theta}_i \tag{7}$$

where

$$\mathbf{B}_{i-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{R}_{i-1} & \mathbf{1} \end{bmatrix} \tag{8}$$

$$\tilde{\mathbf{p}}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} \text{ for revolute joint} \tag{9}$$
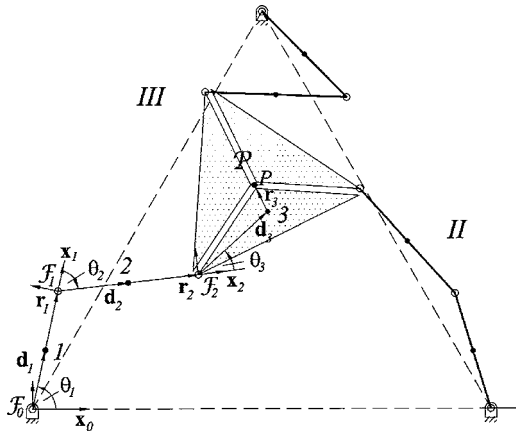
Fig. 2   3-DOF Planar parallel manipulator

$$\tilde{\mathbf{p}}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \text{ for prismatic joint} \qquad (10)$$

where $\mathbf{R}_{i-1}$ is the cross product matrix of $\mathbf{r}_{i-1}$. Further, the twist of link i about mass center C_i is

$$\mathbf{t}_i = \mathbf{B}_i \tilde{\mathbf{t}}_i; \quad \mathbf{B}_i = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{D}_i & \mathbf{1} \end{bmatrix} \qquad (11)$$

where $\mathbf{D}_i$ is the cross product matrix of $\mathbf{d}_i$. Substituting the value of $\tilde{\mathbf{t}}_i$ from Eq. (7) in the above equation we obtain

$$\mathbf{t}_i = \mathbf{B}_i \tilde{\mathbf{B}}_{i-1} \mathbf{t}_{i-1} + \mathbf{B}_i \tilde{\mathbf{p}}_i \dot{\theta}_i \qquad (12)$$

We introduce the notation

$$\mathbf{t}_i = \mathbf{B}_{i,i-1} \mathbf{t}_{i-1} + \mathbf{p}_i \dot{\theta}_i \qquad (13)$$

where

$$\mathbf{B}_{i,i-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{A}_i & \mathbf{1} \end{bmatrix} \qquad (14)$$

$$\mathbf{p}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{D}_i \mathbf{e}_i \end{bmatrix} \text{ for revolute joint} \qquad (15)$$

$$\mathbf{p}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \text{ for prismatic joint} \qquad (16)$$

where $\mathbf{A}_i$ is the cross product matrix of $(\mathbf{r}_{i-1} + \mathbf{d}_i)$. Matrix $\mathbf{B}_{i,i-1}$ may be called the *twist propagation matrix* while $\mathbf{p}_i$, the *twist generator*. The twist $\mathbf{t}_i$ is thus the sum of twist $\mathbf{t}_{i-1}$ and the twist generated at the joint i, both evaluated at C_i. Equation (13) is recursive in nature and is in fact the forward recursion part of the *recursive Newton-Euler algorithm* proposed by Luh et al. [12].

Modeling of the 3R-Planar Platform Manipulator

Figure 2 shows the 3R planar platform manipulators with three degrees of freedom [20]. For the sake of simplicity we restrict ourselves to system that has: (1) only revolute joints, (2) identical legs; and (3) a moving platform in the shape of an equilateral triangle. The three-degrees-of-freedom (three-dof) planar manipulator consists of three identical dyads, numbered *I*, *II*, and *III* coupling the platform $\mathcal{P}$ with the base, such that their fixed pivots lie on the vertices of an equilateral triangle, as well. The proximal and the distal links of each dyad are numbered 1 and 2, respectively. Joint 1 of each dyad is actuated. The centroidal moment of inertia of each link about the axis normal to the xy plane is I_i, for $i=1,2$. The mass of the platform is given by m_P, its mass center located at P, the centroid of the equilateral triangle, and the centroidal moment of inertia about an axis similarly oriented is I_P.

We divide the manipulator into three serial chains, *I*, *II*, and *III*, by dividing the rigid platform $\mathcal{P}$ into *three parts* such that the operation point of the end effector of *each open chain* lies at point P, the mass center of the platform $\mathcal{P}$. Cutting the platform in this manner is advantageous due to the following reasons:

• Torques may be applied to the joints that otherwise need to be cut to open the chains.
• Joint friction may be accommodated directly for such joints.
• Cutting the links (platform) produces a more streamlined recursive kinematic and dynamic modeling for parallel manipulators.

The first two advantages are discussed in greater detail in Yiu et al. [21]; we will discuss these issues in detail in the ensuing analysis.

Recursive Forward Kinematics. The forward kinematics problem for a parallel manipulator is defined as: Given the actuated-joint angles, velocities, and accelerations, find the position, twists, and twist rates of the platform and all the other links.

Position Analysis. The displacement analysis is a critical first step and we adopt the approach proposed by Ma and Angeles [22] to this end.

Velocity Analysis. Since the manipulator is planar, we use two-dimensional position vectors, three-dimensional twist vector $\mathbf{t} = [\omega, \mathbf{v}]^T$, and three-dimensional wrench vectors $\mathbf{w} = [n, \mathbf{f}]^T$, where ω is the angular velocity, $\mathbf{v}$ is the two-dimensional velocity vector, n is the angular moment, and $\mathbf{f}$ is the two-dimensional force vector. For *each chain*, we define position vectors $\mathbf{d}_i$ from the ith joint axis to the mass center of link i, $\mathbf{r}_i$ from mass center of link i to the $(i+1)$st joint axis, and $\mathbf{a}_i = \mathbf{d}_i + \mathbf{r}_i$ as shown in Fig. 2.

The twist of the end effector of *any chain* is given by Saha and Schiehlen [17] as

$$\mathbf{t}_P = \mathbf{B}_{P3} \mathbf{t}_3 \qquad (17)$$

where

$$\mathbf{B}_{P3} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{Er}_3 & \mathbf{1} \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad$$

$\mathbf{t}_3$, the twist of the third link with respect to its mass center, is computed recursively for *each serial chain* from its preceding link as

$$\mathbf{t}_3 = \mathbf{B}_{32} \mathbf{t}_2 + \mathbf{p}_3 \dot{\theta}_3 \qquad (18)$$

$$\mathbf{B}_{32} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{E}(\mathbf{r}_2 + \mathbf{d}_3) & \mathbf{1} \end{bmatrix}; \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ \mathbf{Ed}_3 \end{bmatrix}$$

where the 3×3 matrix $\mathbf{B}_{32}$ is the *twist-propagation matrix* and $\mathbf{p}_3$ is the *twist generator*, $\mathbf{t}_2$ is the twist of link 2 with respect to its mass center; $\dot{\theta}_3$ is the relative rate of the third joint, while $\mathbf{0}$ is the two-dimensional zero vector and $\mathbf{1}$ is the 2×2 identity matrix.

A useful relation is first introduced which will be exploited in the ensuing analysis. Let $\mathbf{a} = \mathbf{b}x + \mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three-dimensional vectors, while x is a scalar; we may determine the value of the scalar as

$$x = \frac{\mathbf{b}^T}{\mathbf{b}^T \mathbf{b}} (\mathbf{a} - \mathbf{c}) \qquad (19)$$

Substituting this value for x back in $\mathbf{a} = \mathbf{b}x + \mathbf{c}$ and rearranging terms we may eliminate x from the equation to obtain a relationship between $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ alone as

$$\left(\mathbf{1} - \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T \mathbf{b}} \right) \mathbf{a} = \left(\mathbf{1} - \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T \mathbf{b}} \right) \mathbf{c} \qquad (20)$$

where $\mathbf{1}$ is the 3×3 identity matrix, which is the relation sought. We employ this process to eliminate the unactuated joint veloci-

ties from the recursive kinematic equations. Substituting $\mathbf{t}_3$ into Eq. (17), we obtain

$$\mathbf{t}_P = \mathbf{B}_{P3}(\mathbf{B}_{32}\mathbf{t}_2 + \mathbf{p}_3\dot{\theta}_3) \qquad (21)$$

The above equation is solved for $\dot{\theta}_3$

$$\dot{\theta}_3 = \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P2}\mathbf{t}_2) \qquad (22)$$

where the three-dimensional vector $\widetilde{\mathbf{p}}_3$ is defined as $\widetilde{\mathbf{p}}_3 = \mathbf{B}_{P3}\mathbf{p}_3$ and $\delta_3 = \widetilde{\mathbf{p}}_3^T\widetilde{\mathbf{p}}_3$. Therefore, when we finally substitute $\dot{\theta}_3$ into Eq. (21) we obtain

$$\mathbf{\Phi}_3\mathbf{t}_P = \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{t}_2 \qquad (23)$$

where $\mathbf{\Phi}_3 = \mathbf{1} - \widetilde{\mathbf{p}}_3\widetilde{\mathbf{p}}_3^T/\delta_3$ and the property $\mathbf{B}_{P3}\mathbf{B}_{32} = \mathbf{B}_{P2}$ has been used. Again, the twist of link 2 is then computed recursively from the twist of link 1 as

$$\mathbf{t}_2 = \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2$$

$$\mathbf{B}_{21} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{E}(\mathbf{r}_1 + \mathbf{d}_2) & \mathbf{1} \end{bmatrix}; \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ \mathbf{Ed}_2 \end{bmatrix}$$

where $\mathbf{t}_1$ is the twist of link 1 with respect to its mass center, $\dot{\theta}_2$ is the relative angular joint velocity of the second joint, $\mathbf{0}$ is the two-dimensional zero vector, and $\mathbf{1}$ is the 2×2 identity matrix. Substituting $\mathbf{t}_2$ into Eq. (23), we obtain

$$\mathbf{\Phi}_3\mathbf{t}_P = \mathbf{\Phi}_3\mathbf{B}_{P2}(\mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2) \qquad (24)$$

Solving for $\dot{\theta}_2$ we obtain

$$\dot{\theta}_2 = \frac{\widetilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) \qquad (25)$$

where $\widetilde{\mathbf{p}}_2 = \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{p}_2$ and $\delta_2 = \widetilde{\mathbf{p}}_2^T\widetilde{\mathbf{p}}_2$. Substituting $\dot{\theta}_2$ into Eq. (24) leads to

$$\mathbf{\Phi}_2\mathbf{t}_P = \mathbf{\Phi}_2\mathbf{B}_{P1}\mathbf{t}_1 \qquad (26)$$

where the 3×3 matrix $\mathbf{\Phi}_2$ is defined as $\mathbf{\Phi}_2 = \mathbf{\Phi}_3 - \widetilde{\mathbf{p}}_2\widetilde{\mathbf{p}}_2^T/\delta_2$ and the properties $\mathbf{B}_{P2}\mathbf{B}_{21} = \mathbf{B}_{P1}$ and $\mathbf{\Phi}_3^T\mathbf{\Phi}_3 = \mathbf{\Phi}_3$ have been used. Noting the similarity between Eqs. (23) and (26), the kinematics relationships may be written in generic form as

$$\mathbf{\Phi}_i\mathbf{t}_P = \mathbf{\Phi}_i\mathbf{B}_{P,i-1}\mathbf{t}_{i-1} \qquad (27)$$

where $\mathbf{\Phi}_i$ is evaluated recursively as

$$\mathbf{\Phi}_i = \mathbf{\Phi}_{i+1} - \widetilde{\mathbf{p}}_i\widetilde{\mathbf{p}}_i^T/\delta_i \qquad (28)$$

Finally, since joint 1 is actuated, substituting $\mathbf{t}_1 = \mathbf{p}_1\dot{\theta}_1$ into Eq. (26), we can express the twist of the platform $\mathcal{P}$ in terms of $\dot{\theta}_1$ as

$$\mathbf{\Phi}_2\mathbf{t}_P = \mathbf{\Phi}_2\mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1 \qquad (29)$$

This equation illustrates the well-known feature for parallel chains. Note that $\mathbf{\Phi}_2$ is a projection matrix and is thus singular. Next, we write Eq. (29) for *each open chain* to obtain

$$\mathbf{K}\mathbf{t}_P = \mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

$$\text{where} \quad \mathbf{K} = [\mathbf{\Phi}_2^I + \mathbf{\Phi}_2^{II} + \mathbf{\Phi}_2^{III}]:3 \times 3$$

$$\mathbf{\Phi} = [\mathbf{\Phi}_2^I\mathbf{\Phi}_2^{II}\mathbf{\Phi}_2^{III}]:3 \times 9$$

$$\mathbf{B} = \mathrm{diag}(\mathbf{B}_{P1}^I, \mathbf{B}_{P1}^{II}, \mathbf{B}_{P1}^{III}):9 \times 9$$

$$\mathbf{P} = \mathrm{diag}(\mathbf{p}_1^I, \mathbf{p}_1^{II}, \mathbf{p}_1^{III}):9 \times 3$$

$$\dot{\boldsymbol{\theta}}_{ac} = [\dot{\theta}_1^I \dot{\theta}_1^{II} \dot{\theta}_1^{III}]^T$$

where all dimensions have been stated for clarity. Finally when $\mathbf{K}$ is nonsingular,[2] we may solve for the end effector velocity in terms of the actuated velocities as

$$\mathbf{t}_P = \mathbf{K}^{-1}\mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \qquad (30)$$

The unactuated joint velocities of *any chain* may be now be computed by substituting $\mathbf{t}_P$ from Eq. (30) into Eq. (25) to yield

$$\dot{\theta}_2 = \frac{\widetilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{K}^{-1}\mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1) \qquad (31)$$

and substituting $\mathbf{t}_P$, $\mathbf{t}_2$, $\mathbf{t}_1$, and $\dot{\theta}_2$ into Eq. (22)

$$\begin{aligned} \dot{\theta}_3 &= \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}[\mathbf{t}_P - \mathbf{B}_{P2}(\mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2)] \\ &= \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1 - \mathbf{B}_{P2}\mathbf{p}_2\dot{\theta}_2) \\ &= \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}\left[(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) - \mathbf{B}_{P2}\mathbf{p}_2\frac{\widetilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_p - \mathbf{B}_{P1}\mathbf{t}_1)\right] \\ &= \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}\left(1 - \frac{1}{\delta_2}\mathbf{B}_{P2}\mathbf{p}_2\widetilde{\mathbf{p}}_2^T\right)(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1) \end{aligned} \qquad (32)$$

which can be written as

$$\dot{\theta}_3 = \frac{\widetilde{\mathbf{p}}_3^T}{\delta_3}\mathbf{\Psi}_2^T(\mathbf{K}^{-1}\mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} - \mathbf{B}_{P1}\mathbf{p}_1\dot{\theta}_1) \qquad (33)$$

where the 3×3 matrix $\mathbf{\Psi}_2$ is defined as $\mathbf{\Psi}_2 = (\mathbf{1} - \mathbf{B}_{P2}\mathbf{p}_2\widetilde{\mathbf{p}}_2^T/\delta_2)^T$ and $\mathbf{1}$ is the 3×3 identity matrix.

We note that Eqs. (31) and (33) are general and applicable to *each open chain* and that the bracketed term on the right hand side of each equation is the same. This term can be written specifically for *each open chain* as

$$[[\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III}]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

$$[[\mathbf{K}^{-1}\mathbf{\Phi}_2]^I \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III}]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

$$[[\mathbf{K}^{-1}\mathbf{\Phi}_2]^I \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} \quad [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III}]\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

Thus the final relationship between the joint rates and actuated joint rates is expressed in matrix form as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \bar{\mathbf{P}}^I & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{P}}^{II} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \bar{\mathbf{P}}^{III} \end{bmatrix} \begin{bmatrix} \mathbf{L}^I \\ \mathbf{L}^{II} \\ \mathbf{L}^{III} \end{bmatrix} \mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \qquad (34)$$

where the 3×9 matrix $\bar{\mathbf{P}}_i$ is defined as $\bar{\mathbf{P}}_i = [\mathrm{diag}(\widetilde{\mathbf{p}}_1^T/\delta_1, \widetilde{\mathbf{p}}_2^T/\delta_2, \widetilde{\mathbf{p}}_3^T\mathbf{\Psi}_2^T/\delta_3)]^i$, while $\widetilde{\mathbf{p}}_1^i$ as explicitly $\widetilde{\mathbf{p}}_1^i = (\mathbf{B}_{P1}\mathbf{p}_1)^i$ for $i = I$, II, and III, and the 9×9 matrices $\mathbf{L}$ are defined for *each open chain* as

$$\mathbf{L}^I = \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \end{bmatrix}$$

$$\mathbf{L}^{II} = \begin{bmatrix} \mathbf{O} & \mathbf{1} & \mathbf{O} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{III} \end{bmatrix}$$

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[2]See [23] for a more detailed discussion of the type of singularities.

$$\mathbf{L}^{III} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{1} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III} \\ [\mathbf{K}^{-1}\mathbf{\Phi}_2]^I & [\mathbf{K}^{-1}\mathbf{\Phi}_2]^{II} & [\mathbf{K}^{-1}\mathbf{\Phi}_2 - \mathbf{1}]^{III} \end{bmatrix}$$

Equation (34) can be written in compact form as

$$\dot{\boldsymbol{\theta}} = \bar{\mathbf{P}}\mathbf{L}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac} \tag{35}$$

where the 9×27 matrix $\bar{\mathbf{P}}$ is defined as $\bar{\mathbf{P}} = \mathrm{diag}(\bar{\mathbf{P}}^I, \bar{\mathbf{P}}^{II}, \bar{\mathbf{P}}^{III})$ and the 27×9 matrix $\mathbf{L}$ is defined as $\mathbf{L} = [(\mathbf{L}^I)^T \ (\mathbf{L}^{II})^T \ (\mathbf{L}^{III})^T]^T$. Note that, except for $\mathbf{L}$, which is full but still retains a special form, all other matrices are block-diagonal.

Acceleration Analysis. The acceleration terms, for *any chain*, by differentiating Eq. (21) as

$$\dot{\mathbf{t}}_P = \dot{\mathbf{B}}_{P3}\mathbf{t}_3 + \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3) + \mathbf{B}_{P3}\mathbf{p}_3\ddot{\theta}_3 \tag{36}$$

Adopting a process similar to the one discussed for the velocity analysis, we may solve for the unactuated joint acceleration for $\ddot{\theta}_3$ as

$$\ddot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}[\dot{\mathbf{t}}_P - \dot{\mathbf{B}}_{P3}\mathbf{t}_3 - \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3)] \tag{37}$$

Substituting $\ddot{\theta}_3$ back into Eq. (36)

$$\mathbf{\Phi}_3\dot{\mathbf{t}}_P = \mathbf{\Phi}_3[\dot{\mathbf{B}}_{P3}\mathbf{t}_3 + \mathbf{B}_{P3}(\dot{\mathbf{B}}_{32}\mathbf{t}_2 + \mathbf{B}_{32}\dot{\mathbf{t}}_2 + \dot{\mathbf{p}}_3\dot{\theta}_3)] \tag{38}$$

Now we obtain the expression for $\dot{\mathbf{\Phi}}_3$. Substituting $\mathbf{t}_3$ and $\dot{\theta}_3$ into Eq. (38) and rearranging leads to

$$\mathbf{\Phi_3}\dot{\mathbf{t}}_P = \mathbf{\Phi_3}\mathbf{B}_{P2}\dot{\mathbf{t}}_2 + \mathbf{\Phi_3}\dot{\mathbf{B}}_{P2}\mathbf{t}_2 - \left[\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3}\right]\mathbf{B}_{P2}\mathbf{t}_2$$

$$+ \left[\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3}\right]\mathbf{t}_P \tag{39}$$

where the relation $(\dot{\mathbf{B}}_{P3}\mathbf{B}_{32} + \mathbf{B}_{P3}\dot{\mathbf{B}}_{32}) = \dot{\mathbf{B}}_{P2}$ has been used. Time-differentiating Eq. (23) we obtain:

$$\mathbf{\Phi}_3\dot{\mathbf{t}}_P = \mathbf{\Phi}_3\mathbf{B}_{P2}\dot{\mathbf{t}}_2 + \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2 - \dot{\mathbf{\Phi}}_3\mathbf{t}_P \tag{40}$$

Comparing Eqs. (39) with (40), we obtain

$$\dot{\mathbf{\Phi}}_3 = -\mathbf{\Phi}_3(\dot{\mathbf{B}}_{P3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{p}}_3^T}{\delta_3} \tag{41}$$

or

$$\dot{\mathbf{\Phi}}_3 = -\mathbf{\Phi}_3\frac{\dot{\tilde{\mathbf{p}}}_3\tilde{\mathbf{p}}_3^T}{\delta_3} \tag{42}$$

Now $\mathbf{t}_2 = \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2$, and hence, $\dot{\mathbf{t}}_2 = \dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \mathbf{p}_2\ddot{\theta}_2 + \dot{\mathbf{p}}_2\dot{\theta}_2$. Substituting $\mathbf{t}_2$ and $\dot{\mathbf{t}}_2$ in Eq. (40) we obtain

$$(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_P) = \mathbf{\Phi}_3\mathbf{B}_{P2}\mathbf{p}_2\ddot{\theta}_2 + [\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2)$$

$$+ \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2 + \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2] \tag{43}$$

Solving the above equation for $\ddot{\theta}_2$

$$\ddot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}\{(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_P) - [\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2$$

$$+ \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2]\} \tag{44}$$

substituting $\ddot{\theta}_2$ back into Eq. (43),

$$\bar{\mathbf{\Phi}}_2(\dot{\mathbf{\Phi}}_3\mathbf{t}_P + \mathbf{\Phi}_3\dot{\mathbf{t}}_p) = \bar{\mathbf{\Phi}}_2[\mathbf{\Phi}_3\mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \mathbf{\Phi}_3\dot{\mathbf{B}}_{P2}\mathbf{t}_2$$

$$+ \dot{\mathbf{\Phi}}_3\mathbf{B}_{P2}\mathbf{t}_2] \tag{45}$$

where $\bar{\mathbf{\Phi}}_2 = \mathbf{1} - \tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T/\delta_2$. However, we note that

$$\bar{\mathbf{\Phi}}_2\mathbf{\Phi}_3 = \left(\mathbf{\Phi}_3 - \frac{\tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T}{\delta_2}\right) = \mathbf{\Phi}_2$$

where the relation $\mathbf{\Phi}_3^T\mathbf{\Phi}_3 = \mathbf{\Phi}_3$ was used. Substituting $\bar{\mathbf{\Phi}}_2\mathbf{\Phi}_3 = \mathbf{\Phi}_2$ and rearranging Eqs. (44) and (45) leads to

$$\ddot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}[\mathbf{\Phi}_3(\dot{\mathbf{t}}_P - \mathbf{a}_1) - \mathbf{a}_2] \tag{46}$$

$$\mathbf{\Phi}_2\dot{\mathbf{t}}_P = \mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2 \tag{47}$$

where $\mathbf{a}_1 = \mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \dot{\mathbf{B}}_{P2}\mathbf{t}_2$ and $\mathbf{a}_2 = \dot{\mathbf{\Phi}}_3(\mathbf{B}_{P2}\mathbf{t}_2 - \mathbf{t}_P)$.

Finally, adding Eq. (47) for *each open chain* and solving for $\dot{\mathbf{t}}_p$

$$\dot{\mathbf{t}}_P = \mathbf{K}^{-1}([\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^I + [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{II} + [\mathbf{\Phi}_2\mathbf{a}_1 + \bar{\mathbf{\Phi}}_2\mathbf{a}_2]^{III}) \tag{48}$$

Summary of Forward Kinematics. The overall process of computing the forward kinematics can be summarized as:

1. With the data, calculate $\mathbf{B}_{21}$, $\mathbf{B}_{32}$, $\mathbf{B}_{31}$, $\mathbf{B}_{P3}$, $\mathbf{B}_{P2}$, $\mathbf{B}_{P1}$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. For *each chain*, we calculate

$$\tilde{\mathbf{p}}_3 = \mathbf{B}_{P3}\mathbf{p}_3$$

$$\delta_3 = \tilde{\mathbf{p}}_3^T\tilde{\mathbf{p}}_3$$

$$\mathbf{\Phi}_3 = \left[\mathbf{1} - \frac{\tilde{\mathbf{p}}_3\tilde{\mathbf{p}}_3^T}{\delta_3}\right]$$

$$\tilde{\mathbf{p}}_2 = \mathbf{B}_{P2}\mathbf{p}_2$$

$$\delta_2 = \tilde{\mathbf{p}}_2^T\tilde{\mathbf{p}}_2$$

$$\mathbf{\Phi}_2 = \mathbf{\Phi}_3 - \frac{\tilde{\mathbf{p}}_2\tilde{\mathbf{p}}_2^T}{\delta_2}$$

2. Form matrices $\mathbf{K}$, $\mathbf{\Phi}$, $\mathbf{B}$, and $\mathbf{P}$ with values received from *each chain* and calculate the platform twist $\mathbf{t}_P$ from:

$$\mathbf{K}\mathbf{t}_p = \mathbf{\Phi}\mathbf{B}\mathbf{P}\dot{\boldsymbol{\theta}}_{ac}$$

3. Obtain the twists and joint rates recursively for *each chain*, using $\mathbf{t}_P$.

$$\mathbf{t}_1 = \mathbf{p}_1\dot{\theta}_1$$

$$\dot{\theta}_2 = \frac{\tilde{\mathbf{p}}_2^T}{\delta_2}(\mathbf{t}_P - \mathbf{B}_{P1}\mathbf{t}_1)$$

$$\mathbf{t}_2 = \mathbf{B}_{21}\mathbf{t}_1 + \mathbf{p}_2\dot{\theta}_2$$

$$\dot{\theta}_3 = \frac{\tilde{\mathbf{p}}_3^T}{\delta_3}(\mathbf{t}_P - \mathbf{B}_{P2}\mathbf{t}_2)$$

$$\mathbf{t}_3 = \mathbf{B}_{32}\mathbf{t}_2\mathbf{p}_3\dot{\theta}_3$$

Now calculate twist rates and joint accelerations for *each chain*. First, calculate, i.e., $\dot{\mathbf{B}}_{21}$, $\dot{\mathbf{B}}_{32}$, $\dot{\mathbf{B}}_{31}$, $\dot{\mathbf{B}}_{P3}$, $\dot{\mathbf{B}}_{P2}$, $\dot{\mathbf{B}}_{P1}$, $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$, and $\dot{\mathbf{p}}_3$ using joint velocities calculated above

$$\dot{\mathbf{t}}_1 = \mathbf{p}_1\ddot{\theta}_1 + \dot{\mathbf{p}}_1\dot{\theta}_1$$

$$\mathbf{a}_1 = \mathbf{B}_{P2}(\dot{\mathbf{B}}_{21}\mathbf{t}_1 + \mathbf{B}_{21}\dot{\mathbf{t}}_1 + \dot{\mathbf{p}}_2\dot{\theta}_2) + \dot{\mathbf{B}}_{P2}\mathbf{t}_2$$

$$\dot{\mathbf{\Phi}}_3 = -\mathbf{\Phi}_3(\dot{\mathbf{B}}_{p3}\mathbf{p}_3 + \mathbf{B}_{P3}\dot{\mathbf{p}}_3)\frac{\tilde{\mathbf{P}}_3^T}{\delta_3}$$

$$a_2 = \dot{\Phi}_3(B_{P2}t_2 - t_P)$$

$$\bar{\Phi}_2 = 1 - \frac{\tilde{p}_2 \tilde{p}^T}{\delta_2}$$

4. Using Φ_2, $\bar{\Phi}_2$, a_1, and a_2 from each chain calculate $\dot{t}_P$ from

$$K\dot{t}_P = [\Phi_2 a_1 + \bar{\Phi}_2 a_2]^I + [\Phi_2 a_1 + \bar{\Phi}_2 a_2]^{II} + [\Phi_2 a_1 + \bar{\Phi}_2 a_2]^{III}$$

5. Calculate joint accelerations and twist rates for *each chain*

$$\ddot{\theta}_2 = \frac{\tilde{p}_2^T}{\delta_2}[\Phi_3(\dot{t}_P - a_1) - a_2]$$

$$\dot{t}_2 = \dot{B}_{21}t_1 + B_{21}\dot{t}_1\dot{p}_2\dot{\theta}_2 + p_2\ddot{\theta}_2$$

$$\ddot{\theta}_3 = \frac{\tilde{p}_3^T}{\delta_3}[\dot{t}_P - \dot{B}_{P3}t_3 - B_{P3}(\dot{B}_{32}t_2 + B_{32}\dot{t}_2 + \dot{p}_3\dot{\theta}_3)]$$

$$\dot{t}_3 = B_{32}\dot{t}_2 + \dot{B}_{32}t_2 + p_3\dot{\theta}_3\dot{p}_3\ddot{\theta}_3$$

Inverse Dynamics. The inverse-dynamics problem is defined as follows: Given the time-histories of all the system degrees-of-freedom, compute the time histories of the controlling actuated joint torques and forces. As in the case of the kinematics calculations, we again divide the platform into three parts and assign cut sections of platform $\mathcal{P}$ to each open chain. Each cut section thus becomes the "end effector" of the corresponding serial chain. Further, we divide the mass of the platform (including any tool carried by the platform) and assign its corresponding moment of inertia, with respect to the mass center of the platform, to the "end effector" of each chain. Any working wrench applied to the platform has to be appropriately divided in a similar fashion. The Newton-Euler equations for *each open chain* is, thus,

$$M\dot{t} + \dot{M}t = w^A + \underbrace{\bar{w}^W + w^g}_{w^G} + w^C \quad (49)$$

where M is the 9×9 mass matrix, t is the nine-dimensional twist vector of the whole chain, w^A is the wrench applied by the actuators, $\bar{w}^W = [0^T, 0^T, (w^W)^T]^T$, where w^W is the corresponding working wrench applied at the "end effector" of the corresponding chain, w^g is the gravity wrench, and w^C are the constraint wrenches all these being nine-dimensional vectors. The friction forces have been left out for the sake of simplicity but they can be readily taken into account by means of dissipation function [18]. In particular, the twist vector t may now be written as [16].

$$t = N_l N_d \dot{\theta} \quad (50)$$

where $N_l N_d$ is the decoupled orthogonal complement and $\dot{\theta}$ is the joint-rate vector of the chain. For our manipulator, for *each open chain* Eq. (50) is

$$t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & O & O \\ B_{21} & 1 & O \\ B_{31} & B_{32} & 1 \end{bmatrix}}_{N_l} \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}}_{N_d} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \quad (51)$$

where all terms have been previously defined. Now, the constraint wrenches w^C do not develop any power, and hence, $t^T w^C$ is 0; by virtue of Eq. (50), w^C lies in the nullspace of $N_d^T N_l^T$. To eliminate joint constraint wrenches, we premultiply both sides of Eq. (49) by $N_d^T N_l^T$, and noting that, for planar manipulators $\dot{M} = O$

$$N_d^T N_l^T M\dot{t} = \tilde{\tau} + N_d^T N_l^T w^G \quad (52)$$

Time differentiating Eq. (50) and substituting the expression for $\dot{t}$ in Eq. (52)
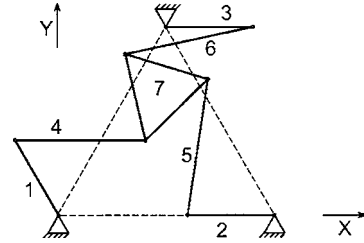
Fig. 3 Three-DOF Planar parallel manipulator used in the example

$$N_d^T N_l^T [MN_l N_d \ddot{\theta} + (M\dot{N}_l N_d + MN_l \dot{N}_d)\dot{\theta}] = \tilde{\tau} + N_d^T N_l^T w^G \quad (53)$$

where $\tilde{\tau} = N_d^T N_l^T w^A$ is the joint torque vector for the chain and is given by $\tilde{\tau} = [\tilde{\tau}_1 \quad \tilde{\tau}_2 \quad \tilde{\tau}_3]^T$ for *each open chain*. We can write the above equation in compact form as

$$I\ddot{\theta} + C\dot{\theta} = \tilde{\tau} + \tau^G \quad (54)$$

where $I = N_d^T N_l^T MN_l N_d$ and $C = N_d^T N_l^T (M\dot{N}_l N_d + MN_l \dot{N}_d)$, I being the *generalized inertia matrix of the chain* and C the matrix of coriolis and centrifugal forces.

Notice that the distribution of the working wrench between chains is not important, as the torques evaluated at this stage are projected onto the minimum actuated joint space in the step that follows. We will discuss the case where the working wrench is assumed to have been distributed evenly among the subsystems.

Projecting Joint Torques onto Minimal-Coordinate Space. As a second step, we write the dynamics equation for *each chain* and couple them with Lagrange multipliers, thereby obtaining the dynamics equation of the whole manipulator as

$$\begin{bmatrix} [I\ddot{\theta} + C\dot{\theta}]^I \\ [I\ddot{\theta} + C\dot{\theta}]^{II} \\ [I\ddot{\theta} + C\dot{\theta}]^{III} \end{bmatrix} = \begin{bmatrix} \tau^I \\ \tau^{II} \\ \tau^{III} \end{bmatrix} + \begin{bmatrix} \tau_G^I \\ \tau_G^{II} \\ \tau_G^{III} \end{bmatrix} - A^T \lambda \quad (55)$$

where A is the *loop-closure constraint Jacobian*, which appears in the constraints in the form

$$A\dot{\theta} = 0$$

Now, by defining $\dot{\theta} = J\dot{\theta}_{ac}$, it is clear that J lies in the nullspace of A and may be called the *loop-closure orthogonal complement*. Premultiplying both sides of Eq. (55) by J^T we obtain

$$J^T \begin{bmatrix} [I\ddot{\theta} + C\dot{\theta} - \tau_G]^I \\ [I\ddot{\theta} + C\dot{\theta} - \tau_G]^{II} \\ [I\ddot{\theta} + C\dot{\theta} - \tau_G]^{III} \end{bmatrix} = \tau_{ac} \quad (56)$$

where τ_{ac} is the vector of actuator torques. Notice that the bracketed terms are nothing else than $\tilde{\tau}^j$, which can be found for *each open chain*, for $j=I$, II, and III, recursively [16]. We may therefore write Eq. (56) as

Table 1 Dimension and inertia properties

Link i	L_i(m)	m_i(kg)	I_i(Kg m2)
1,2,3	0.4	3.0	0.04
4,5,6	0.6	4.0	0.12
7	0.4	8.0	0.0817
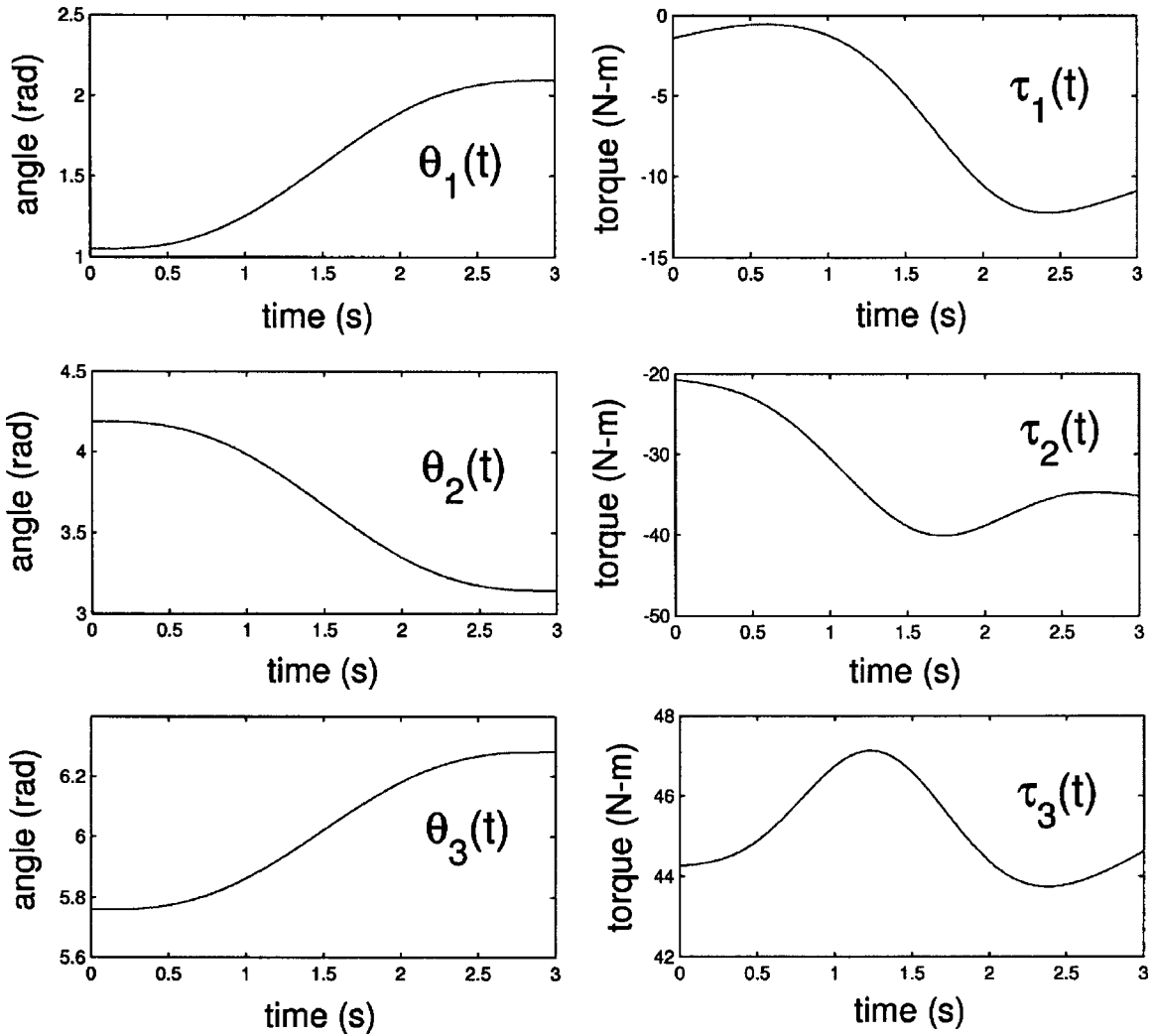
Fig. 4 Desired trajectory and required driving torques

$$\mathbf{J}^T \begin{bmatrix} \widetilde{\boldsymbol{\tau}}^I \\ \widetilde{\boldsymbol{\tau}}^{II} \\ \widetilde{\boldsymbol{\tau}}^{III} \end{bmatrix} = \boldsymbol{\tau}_{AC} \tag{57}$$

which is the relation sought, where $\mathbf{J}$, given by Eq. (35), may be written in a slightly different form

$$\mathbf{J} = \bar{\mathbf{P}} \mathbf{L} \widetilde{\mathbf{P}} \tag{58}$$

where $\quad \mathbf{L} = \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} \\ (\mathbf{S}^I - \mathbf{1}) & \mathbf{S}^{II} & \mathbf{S}^{III} \\ (\mathbf{S}^I - \mathbf{1}) & \mathbf{S}^{II} & \mathbf{S}^{III} \\ \mathbf{O} & \mathbf{1} & \mathbf{O} \\ \mathbf{S}^I & (\mathbf{S}^{II} - \mathbf{1}) & \mathbf{S}^{III} \\ \mathbf{S}^I & (\mathbf{S}^{II} - \mathbf{1}) & \mathbf{S}^{III} \\ \mathbf{O} & \mathbf{O} & \mathbf{1} \\ \mathbf{S}^I & \mathbf{S}^{II} & (\mathbf{S}^{III} - \mathbf{1}) \\ \mathbf{S}^I & \mathbf{S}^{II} & (\mathbf{S}^{III} - \mathbf{1}) \end{bmatrix},$

$$\widetilde{\mathbf{P}} = \begin{bmatrix} \widetilde{\mathbf{p}}_1^I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{p}}_1^{II} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{p}}_1^{III} \end{bmatrix}$$

and $\mathbf{S}^j = \mathbf{K}^{-1} \boldsymbol{\Phi}_2^j$ for $j = I, II,$ and III. Substituting $\mathbf{J}$ into Eq. (57) and rearranging we obtain the actuated torques as

$$\tau_1^I = (\widetilde{\mathbf{p}}_1^I)^T [(\mathbf{S}^I)^T (\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) + \hat{\mathbf{p}}_1^I - \hat{\mathbf{p}}_2^I - \hat{\mathbf{p}}_3^I]$$

$$\tau_1^{II} = (\widetilde{\mathbf{p}}_1^{II})^T [(\mathbf{S}^{II})^T (\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) + \hat{\mathbf{p}}_1^{II} - \hat{\mathbf{p}}_2^{II} - \hat{\mathbf{p}}_3^{II}]$$

$$\tau_1^{III} = (\widetilde{\mathbf{p}}_1^{III})^T [(\mathbf{S}^{III})^T (\hat{\mathbf{p}}_2^I + \hat{\mathbf{p}}_3^I + \hat{\mathbf{p}}_2^{II} + \hat{\mathbf{p}}_3^{II} + \hat{\mathbf{p}}_2^{III} + \hat{\mathbf{p}}_3^{III}) + \hat{\mathbf{p}}_1^{III} - \hat{\mathbf{p}}_2^{III} - \hat{\mathbf{p}}_3^{III}]$$

where $\hat{\mathbf{p}}_k^j = [\widetilde{\tau}_k \boldsymbol{\Psi}_{k-1} \widetilde{\mathbf{p}}_k / \delta_k]^j$ for $k = 1, 2,$ and 3 and $j = I, II,$ and III, where $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_1$ are the three-dimensional identity matrices. But $\widetilde{\mathbf{p}}_1^T \hat{\mathbf{p}}_1 = \widetilde{\tau}_1$ and hence the above equation set is written finally as

$$\boldsymbol{\tau}_{\mathbf{ac}} = \begin{bmatrix} \widetilde{\boldsymbol{\tau}}_1^I + (\widetilde{\mathbf{p}}_1^I)^T [(\mathbf{S}^I)^T (\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^I] \\ \widetilde{\boldsymbol{\tau}}_1^{II} + (\widetilde{\mathbf{p}}_1^{II})^T [(\mathbf{S}^{II})^T (\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{II}] \\ \widetilde{\boldsymbol{\tau}}_1^{III} + (\widetilde{\mathbf{p}}_1^{III})^T [(\mathbf{S}^{III})^T (\bar{\mathbf{p}}^I + \bar{\mathbf{p}}^{II} + \bar{\mathbf{p}}^{III}) - \bar{\mathbf{p}}^{III}] \end{bmatrix}$$

where $\bar{\mathbf{p}} = \hat{\mathbf{p}}_2 + \hat{\mathbf{p}}_3$ for corresponding chains.

Summary of Inverse Dynamics. To summarize the process of computation of the inverse dynamics

1. From Eq. (52) it is clear that

$$\widetilde{\boldsymbol{\tau}} = \mathbf{N}_d^T \mathbf{N}_i^T (\mathbf{M}\dot{\mathbf{t}} + \mathbf{w}^G)$$

which can be calculated recursively for each chain as follows

$$\boldsymbol{\gamma}_3 = (\mathbf{M}_3 \dot{\mathbf{t}}_3 + \mathbf{w}_3^G)$$

$$\boldsymbol{\gamma}_2 = (\mathbf{M}_2 \dot{\mathbf{t}}_2 + \mathbf{w}_2^G) + \mathbf{B}_{32}^T \boldsymbol{\gamma}_3$$

$$\boldsymbol{\gamma}_1 = (\mathbf{M}_1 \dot{\mathbf{t}}_1 + \mathbf{w}_1^G) + \mathbf{B}_{21}^T \boldsymbol{\gamma}_2$$

$$\widetilde{\tau}_3 = \mathbf{p}_3^T \boldsymbol{\gamma}_3$$

$$\widetilde{\tau}_2 = \mathbf{p}_2^T \boldsymbol{\gamma}_2$$

$$\widetilde{\tau}_1 = \mathbf{p}_1^T \boldsymbol{\gamma}_1$$

Now we calculate $\overline{\mathbf{p}}$

$$\overline{\mathbf{p}} = \widetilde{\tau}_2 \frac{\overline{\mathbf{p}}_2}{\delta_2} + \widetilde{\tau}_3 \frac{\boldsymbol{\Psi}_2 \overline{\mathbf{p}}_3}{\delta_3}$$

2. Add all $\overline{\mathbf{p}}$ from each chain
3. Calculate actuated joint torques

   Chain *I*:
   $$\tau_1^I = \widetilde{\boldsymbol{\tau}}_1^I + (\overline{\mathbf{p}}_1^I)^T [(\mathbf{S}^I)^T (\overline{\mathbf{p}}^I + \overline{\mathbf{p}}^{II} + \overline{\mathbf{p}}^{III}) - \overline{\mathbf{p}}^I]$$

   Chain *II*:
   $$\tau_1^{II} = \widetilde{\boldsymbol{\tau}}_1^{II} + (\widetilde{\mathbf{p}}_1^{II})^T [(\mathbf{S}^{II})^T (\overline{\mathbf{p}}^I + \overline{\mathbf{p}}^{II} + \overline{\mathbf{p}}^{III}) - \overline{\mathbf{p}}^{II}]$$

   Chain *III*:
   $$\tau_1^{III} = \widetilde{\boldsymbol{\tau}}_1^{III} + (\widetilde{\mathbf{p}}_1^{III})^T [(\mathbf{S}^{III})^T (\overline{\mathbf{p}}^I + \overline{\mathbf{p}}^{II} + \overline{\mathbf{p}}^{III}) - \overline{\mathbf{p}}^{III}]$$

Example

Parameters and Initial Conditions. We use the same parameters for the 3R planar Stewart-Gough platform shown in Fig. 3 as in Ma and Angeles [22]. The end effector, labeled 7, has the shape of an equilateral triangle, with sides of length l_7, links 1, 2, and 3 have a length l_1, links 4, 5, and 6 have length l_4, and the three fixed revolute joints form an equilateral triangle with sides of length l_0. The prescribed motion drivers given by Ma and Angeles [22] were

$$\theta_1 = \frac{1}{3}\pi + \frac{1}{6}\left(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}\right)$$

$$\theta_2 = \frac{4}{3}\pi + \frac{1}{6}\left(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}\right)$$

$$\theta_3 = \frac{11}{6}\pi + \frac{1}{12}\left(\frac{2\pi t}{T} - \sin\frac{2\pi t}{T}\right)$$

where $T = 3$ s. However, since these initial conditions are not sufficient to define a unique initial posture of the manipulator, we use the initial configuration given by Geike and McPhee [24]

$$\theta_1 = \frac{1}{3}\pi \text{ rad} \quad \theta_4 = -0.865 \text{ rad} \quad x_7 = 0.728 \text{ m}$$

$$\theta_2 = \frac{4}{3}\pi \text{ rad} \quad \theta_5 = -2.102 \text{ rad} \quad y_7 = 0.233 \text{ m}$$

$$\theta_3 = \frac{11}{6}\pi \text{ rad} \quad \theta_6 = -0.976 \text{ rad} \quad \theta_7 = 3.916 \text{ rad}$$

The parameters of the manipulator are given in Table 1, gravity acts in the $-Y$ direction.

Results. We first perform the inverse dynamics in order to compute a time history of actuation forces that would realize the prescribed motions. Using the above parameters, the resulting torques for the actuated joints 1, 2, and 3 were evaluated using the inverse dynamics model discussed in this paper. The resulting set of torques which realize the prescribed motions is shown in Fig. 4, which tally with those given by Ma and Angeles [22].

Conclusion

A modular and recursive inverse dynamics algorithm based on decoupled natural orthogonal complement was presented for parallel architecture manipulators. In particular, these results were illustrated in the context of the formulation and subsequent computation of inverse dynamics for a planar 3R parallel manipulator. The modularity is achieved by projecting the set of submodel dynamics onto the space of feasible motions, taking advantage of the decoupled natural orthogonal complement for this projection. The recursive computation of DeNOC thus leads to the recursive nature of the suggested algorithm.

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