Denavit-Hartenberg Parameterization of Euler Angles

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Euler angles describe rotations of a rigid body in three-dimensional Cartesian space, as can be obtained by, say, a spherical joint. The rotation carried out by a spherical joint can also be expressed by using three intersecting revolute joints that can be described using the popular Denavit-Hartenberg (DH) parameters. However, the motions of these revolute joints do not necessarily correspond to any set of the Euler angles. This paper attempts to correlate the Euler angles and DH parameters by introducing a concept of DH parameterization of Euler angels. A systematic approach is presented in order to obtain the DH parameters for any Euler angles set. This gives rise to the concept of Euler-angle-joints (EAJs), which provide rotations equivalent to a particular set of Euler angles. Such EAJs can be conveniently used for the modeling of multibody systems having multiple-degrees-of-freedom joints. [DOI: 10.1115/1.4005467]

Keywords: Euler angles, DH parameterization, intersecting revolute joints, Euler-angle-joints (EAJs)

1 Introduction

Rotation representation of a rigid body moving in a three-dimensional Cartesian space is important for obtaining its kinematic and dynamic behavior. Selection of appropriate coordinates is vital, particularly, with multiple-degree-of-freedom (multiple-DOF) joints connecting two neighboring links or bodies. Many schemes [1–3] are available to represent a rotation in space. The representation using the rotation matrix of nine direction cosines is one such scheme; however, it is not preferred by many as it uses dependent coordinates. The use of Euler angles [1] is an alternative choice. It has wide acceptability in the fields of aerospace, biomechanics, automobile, and others due to its independent representation. For spatial rotations, one may also use other minimal set representations [2] like Bryant (or Cardan) angles, Rodriguez parameters, etc. It is worth mentioning that the fundamental difference between the Euler and Bryant angles lies in the fact that the former represents a sequence of rotations about the same axis separated with a rotation about a different axis, denoted as α–β–α, whereas the latter represents the sequence of rotations about three different axes, denoted as α–β–γ. They are also commonly referred to as symmetric and asymmetric sets of Euler angles in the literature [2]. For convenience, the name Euler angles in this work generally refers to both Euler and Bryant angles, hereafter. The use of Euler parameters [3] is another popular choice and uses four parameters though the DOF of a rigid body rotation in space is three.

On the other hand, a multiple-DOF joint of a robotic system connecting a pair of links is treated as a combination of one-DOF joints, e.g., revolute or prismatic joints [4,5]. Robotic systems such as humanoid, legged robot, robotic hand, etc. contain multiple-DOF joints, say, a universal or spherical joint. A universal joint, also known as Hooke's joint, is a combination of two orthogonally intersecting revolute joints. Similarly, the kinematic behavior of a spherical joint can be simulated by a combination of three revolute joints whose axes intersect at a point. These joint axes are typically represented using the well-known Denavit and Hartenberg parameters [6]. It is pointed out here that the rotations carried out by these intersecting revolute joints do not necessarily represent the Euler angle rotations. Hence, it would be interesting to find a correlation between them as both are extensively used in literature for the dynamics of robotic and multibody systems.

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In this paper, an attempt is made to correlate them by introducing the concept of DH parameterization of the Euler angles. A systematic method has been proposed to correlate single axis rotations and the DH representation of the axes of rotations to define the Euler angles of a spatial rotation provided by, say, spherical joint. Such correlation has never been attempted before, at least it is not found in the existing literature. Hence, this forms the fundamental contribution of this paper leading to a novel concept of Euler-angle-joints. EAJs have the specific advantages that they (1) help in obtaining correlation between the Euler angles and the DH parameters, (2) allow one to obtain Euler angle rotations even though the configuration of a link connected by a joint is defined using a set of DH parameters, and (3) make a unified representation of multiple-DOF joints.

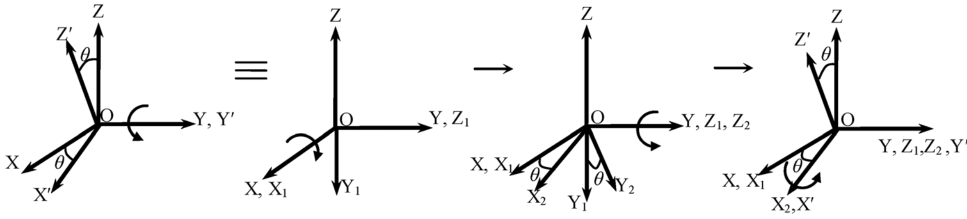
This paper is organized as follows: The DH parameterization of Euler angles is presented in Sec. 2, and the concept of Euler-angle-joints is introduced in Sec. 3. Important characteristic of EAJs are shown in Sec. 4. Finally, conclusions are given in Sec. 5.

2 DH Parameterization of Euler Angles

A major challenge in identifying the Euler/Bryant angles using DH parameters is that, under the DH parameter scheme, the variable rotations always occur about Z axis, whereas the Euler angles are defined by rotation about all the three axes, namely, X, Y, and Z, as shown in Appendix A.1. In the definition of DH parameters (Appendix A.2), (1) a variable rotation about the Z axis denotes the joint angle θ, and (2) a constant rotation about the X axis represents the twist angle α. As a result, the first step towards the DH parameterization of the Euler angles is to represent any Euler angle rotation with respect to Z or X axis only. More specifically, the variable Euler angle rotation has to be always about the Z axis.

Hence, in this section, first each elementary rotation is obtained by the rotation about the Z axis only. It is shown next that any two composite rotations, say, about the Y followed by about the Z, can be shown to be equal to a combination of elementary rotations obtained. Similarly, any set of Euler angles can be obtained by appropriate combination of an elementary rotation followed by a composite rotation or vice versa.

2.1 Elementary Rotations. In the definition of DH parameters as the variable rotation is about the Z axis, any elementary rotation about the Z axis does not require additional transformation. On the other hand, elementary rotations about the X or Y axis must be equivalently rotated about the Z axis. The concept is



(a) (b) $\mathbf{Q}_{X(-90)}$ (c) $\mathbf{Q}_{Z(\theta)}$ (d) $\mathbf{Q}_{X(90)}$

Fig. 1 Rotation about the Y axis

Table 1 Equivalent rotation matrices for elementary rotations

Elementary rotations	Equivalent rotation matrix
Z axis	$\mathbf{Q}_Z = \mathbf{Q}_\theta$
Y axis	$\mathbf{Q}_Y = \mathbf{Q}_{-X}\mathbf{Q}_\theta\mathbf{Q}_{+X}$
X axis	$\mathbf{Q}_X = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_\theta\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$

$\mathbf{Q}_{+Z} \equiv \mathbf{Q}_{Z(90)}, \mathbf{Q}_{+X} \equiv \mathbf{Q}_{X(90)}, \mathbf{Q}_\theta \equiv \mathbf{Q}_{Z(\theta)}, \mathbf{Q}_{-X} \equiv \mathbf{Q}_{X(-90)}$, and $\mathbf{Q}_{-Z} \equiv \mathbf{Q}_{Z(-90)}$.

illustrated below with the elementary rotation about the Y axis as shown in Fig. 1(*a*). In order to represent the same elementary rotation using an equivalent rotation about the Z axis, the Z axis of Fig. 1(*a*) has to be first brought parallel to the Y axis before the desired rotation is applied. This can be done by rotating frame O-XYZ about the X axis by $-90°$, as shown in Fig. 1(*b*). The new frame is indicated with O-$X_1Y_1Z_1$. The rotation is indicated with $Q_{X(-90)}$. The desired rotation by an angle θ is now imparted about the Z_1 axis, as shown in Fig. 1(*c*), where the corresponding rotation is indicated by $Q_{Z(\theta)}$. The new frame is O-$X_2Y_2Z_2$. Finally, to take care of the initial rotation about the X axis by $-90°$, an opposite rotation about the X_2 axis is applied, which is indicated in Fig. 1(*d*) by $Q_{X\ (90)}$. The final frame is O-X′Y′Z′. The resultant of the three elementary rotations is the desired rotation about the Y axis, which is given by

$$\mathbf{Q}_Y = \mathbf{Q}_{-X}\mathbf{Q}_\theta\mathbf{Q}_{+X} \tag{1}$$

where for brevity, $\mathbf{Q}_\theta \equiv \mathbf{Q}_{Z(\theta)}, \mathbf{Q}_{-X} \equiv \mathbf{Q}_{X(-90)}, \mathbf{Q}_{+X} \equiv \mathbf{Q}_{X(90)}$ are used. Interestingly, Eq. (1) represents the rotation matrix for rotation about the Y axis in terms of rotation matrices for rotation about the X and Z axes. Hence, the matrix representation in Eq. (1) will be referred to as the equivalent rotation matrix.

One can similarly find an equivalent rotation matrix for the elementary rotation about the X axis. Table 1 shows the equivalent rotation matrix for all three elementary rotations.

2.2 Composite Rotations.

Resultant of two elementary rotations, say, first about the Y axis followed by about the Z axis, is referred to here as the composite rotation YZ. The corresponding rotation matrix is denoted with $\mathbf{Q}_{\mathrm{YZ}}$ and can be obtained using the equivalent rotations matrices $\mathbf{Q}_{\mathrm{Y}}$ and $\mathbf{Q}_{\mathrm{Z}}$, given in Table 1 as

$$\mathbf{Q}_{YZ} = \mathbf{Q}_Y\mathbf{Q}_Z = \overbrace{\mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+X}}^{Y}\overbrace{\mathbf{Q}_{\theta_2}}^{Z} \tag{2}$$

where $\mathbf{Q}_{\theta_1}$ and $\mathbf{Q}_{\theta_2}$ represent the rotations about the Z axis by angles θ_1 and θ_2, and $\mathbf{Q}_{\pm X}$ represents the rotation about the X axis by ± 90. It is interesting to point out that the equivalent rotation matrix $\mathbf{Q}_{ZY}$ due to the sequence of rotations about the Z axis followed by about the Y axis can be obtained by using the transpose rule of the matrix multiplications, i.e.,

$$\mathbf{Q}_{ZY} = \mathbf{Q}_{YZ}^T = \mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X} \tag{3}$$

where $\mathbf{Q}_{\theta_1}$ actually represents $\mathbf{Q}_{\theta_2}^T (= \mathbf{Q}_{-\theta_2})$ in which -θ_2 is substituted by θ_1 as it is the first joint rotation. The expression for $\mathbf{Q}_{ZY}$ can also be verified independently using the equivalent rotations matrices $\mathbf{Q}_Z$ and $\mathbf{Q}_Y$ given in Table 1. The equivalent rotation matrices for the other composite rotations can be similarly obtained, as shown in Table 2.

2.3 Euler Angle Sets.

The elementary and composite rotations obtained in Sec. 2.1 and Sec. 2.2 form the foundation to obtain the equivalent rotation matrices for different Euler angle sets.

2.3.1 ZYZ Euler Angles. The equivalent rotation matrix for the ZYZ Euler angles set can be obtained as a combination of the equivalent rotation matrices due to elementary rotation about Z and the composite rotation about YZ given in Table 1 and Table 2, respectively. The resulting matrix is denoted as $\mathbf{Q}_{\mathrm{ZYZ}}$, and given by

$$\mathbf{Q}_{ZYZ} = \mathbf{Q}_Z\mathbf{Q}_{YZ} = \overbrace{\mathbf{Q}_{\theta_1}}^{Z}\overbrace{\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}}^{YZ} \tag{4}$$

Interestingly, Eq. (4) can also be obtained by combining the equivalent rotation matrices due to composite rotations ZY and the elementary rotation about the Z axis as

$$\mathbf{Q}_{ZYZ} = \mathbf{Q}_{ZY}\mathbf{Q}_Z = \overbrace{\mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}}^{ZY}\overbrace{\mathbf{Q}_{\theta_3}}^{Z} \tag{5}$$

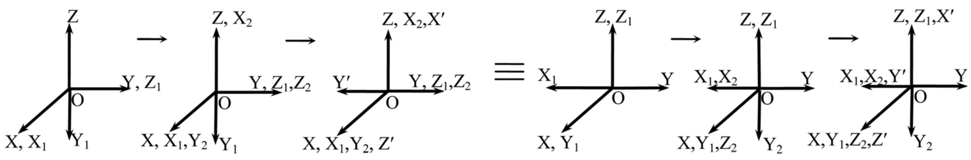
In Eq. (5), $\mathbf{Q}_{ZYZ}$ is nothing but the one obtained in Eq. (4). Therefore, $\mathbf{Q}_{\alpha\beta\gamma}$ is associative in nature provided the sequence of rotations is maintained, i.e.,

$$\mathbf{Q}_{\alpha\beta\gamma} = \mathbf{Q}_{\alpha\beta}\mathbf{Q}_\gamma = \mathbf{Q}_\alpha\mathbf{Q}_{\beta\gamma} \tag{6}$$

Table 2 Equivalent rotation matrices for composite rotations

Composite rotation	Equivalent rotation matrix	Composite rotation	Equivalent rotation matrix
ZX	$\mathbf{Q}_{ZX} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$	ZY	$\mathbf{Q}_{ZY} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}$
YX	$\mathbf{Q}_{YX} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{+Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$	YZ	$\mathbf{Q}_{YZ} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}$
XY	$\mathbf{Q}_{XY} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}$	XZ	$\mathbf{Q}_{XZ} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_2}$

$\mathbf{Q}_{+Z} \equiv \mathbf{Q}_{Z(90)}, \mathbf{Q}_{+X} \equiv \mathbf{Q}_{X(90)}, \mathbf{Q}_{\theta_i} \equiv \mathbf{Q}_{Z(\theta_i)}, \mathbf{Q}_{-X} \equiv \mathbf{Q}_{X(-90)}$, and $\mathbf{Q}_{-Z} \equiv \mathbf{Q}_{Z(-90)}$.



(a) $\mathbf{Q}_{X(-90)}$ (b) $\mathbf{Q}_{Z(-90)}$ (c) $\mathbf{Q}_{X(-90)}$ $\equiv$ (d) $\mathbf{Q}_{Z(-90)}$ (e) $\mathbf{Q}_{X(-90)}$ (f) $\mathbf{Q}_{Z(-90)}$

Fig. 2 Equivalent transformations

Table 3 Equivalent Rotation matrices for Euler angle sets

Euler angles	Equivalent rotation matrix	Euler angles	Equivalent rotation matrix
ZXZ	$\mathbf{Q}_{ZXZ} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_3}$	ZXY	$\mathbf{Q}_{ZXY} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_3}\mathbf{Q}_{+X}$
ZYZ	$\mathbf{Q}_{ZYZ} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}$	ZYX	$\mathbf{Q}_{ZYX} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{+Z}\mathbf{Q}_{\theta_3}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$
YXY	$\mathbf{Q}_{YXY} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_3}\mathbf{Q}_{+X}$	YXZ	$\mathbf{Q}_{YXZ} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{+Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_3}$
YZX	$\mathbf{Q}_{YZX} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$	YZY	$\mathbf{Q}_{YZY} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_3}\mathbf{Q}_{+X}$
XYX	$\mathbf{Q}_{XYX} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{+Z}\mathbf{Q}_{\theta_3}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$	XYZ	$\mathbf{Q}_{XYZ} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}$
XZX	$\mathbf{Q}_{XZX} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$	XZY	$\mathbf{Q}_{XZY} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_3}\mathbf{Q}_{+X}$

$\mathbf{Q}_{+Z} \equiv \mathbf{Q}_{Z(90)}, \mathbf{Q}_{+X} \equiv \mathbf{Q}_{X(90)}, \mathbf{Q}_{\theta_i} \equiv \mathbf{Q}_{Z(\theta_i)}, \mathbf{Q}_{-X} \equiv \mathbf{Q}_{X(-90)}$, and $\mathbf{Q}_{-Z} \equiv \mathbf{Q}_{Z(-90)}$.

2.3.2 XYZ Euler Angles. Next, the equivalent rotation matrix for the XYZ Euler angles, also known as Bryant angles, is derived as a combination of the equivalent rotation matrix $\mathbf{Q}_X$ and the equivalent rotation matrix $\mathbf{Q}_{YZ}$, obtained in Table 1 and Table 2, respectively, i.e.,

$$\mathbf{Q}_{XYZ} = \mathbf{Q}_X\mathbf{Q}_{YZ} = \overbrace{\mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\underline{\mathbf{Q}_{-X}\mathbf{Q}_{-Z}}}^{X}\overbrace{\underline{\mathbf{Q}_{-X}}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}}^{YZ} \quad (7)$$

Note that the rotation matrix $\mathbf{Q}_{-Z}$, representing the constant rotation about the Z axis by $-90°$, in the middle of two constant rotation matrices $\mathbf{Q}_{-X}$, i.e., $\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}$, calls for an additional set of DH parameters, as the DH notations allows only variable rotation about the Z axis. Interestingly, the term $\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}$ is equivalent to $\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$, as illustrated in Fig. 2. The same can be proven using matrix expressions as well. It is also true for any three sequential rotations by ± 90, i.e., $\mathbf{Q}_{\alpha(\pm 90)}\mathbf{Q}_{\beta(\pm 90)}\mathbf{Q}_{\alpha(\pm 90)} = \mathbf{Q}_{\beta(\pm 90)}\mathbf{Q}_{\alpha(\pm 90)}\mathbf{Q}_{\beta(\pm 90)}$, where α and β are the rotations about the X, Y, or Z axis. Hence, replacing the term $\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}$ in Eq. (7) with $\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}$, one obtains

$$\mathbf{Q}_{XYZ} = \mathbf{Q}_{+Z}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{-Z}\mathbf{Q}_{-X}\mathbf{Q}_{-Z}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3} \quad (8)$$

In Eq. (8), the fixed rotation $\mathbf{Q}_{-Z}$ appears next to $\mathbf{Q}_{\theta_1}$, which represents a variable rotation about the Z axis. Hence, the two rotations can be combined as one rotation about the Z axis by $(\theta_1 - 90)°$. As a result, the requirement of an additional set of DH parameters can be avoided. The equivalent rotation matrices for all the twelve symmetric and asymmetric Euler angle sets were developed, as reported in Table 3. They will be used in the subsequent section to evolve the concept of Euler-Angle-Joints.

3 Euler-Angles-Joints (EAJs)

As discussed earlier, many robotic systems consist of multiple-DOF joints, which may be modeled as a combination of more than one intersecting one-DOF joints. The axes of these joints are commonly identified with DH parameters. For example, a spherical joint can be represented by three intersecting joints, axes of which are described using the DH parameters. However, the rotations about the intersecting axes do not necessarily provide the same rotations as obtained by using any set of Euler angles. Now, if the DH parameters of these intersecting revolute joint-axes are assigned based on the equivalent rotation matrices obtained in Table 3, one would actually obtain the Euler angle rotations. Similarly, if the DH parameters of a universal joint, represented with two intersecting revolute joints, are defined using the equivalent rotation matrices obtained in Table 2, the joint rotations give Euler angles. As these intersecting joints provide Euler angle rotations, they are termed as Euler-Angle-Joints. Hence, the EAJs are formally defined as The intersecting revolute joints whose axes are identified using the definition of DH parameters such that the resulting rotations are equivalent to the one described by a particular set of Euler angles. Such correlations are obtained here for the first time and help in a unified representation of a spatial rotation by spherical joint, a two-DOF rotation by universal joint, and a one-DOF rotation by revolute joint. Hence, an algorithm developer can represent different rotations for kinematic and dynamic analyses of multibody systems with utmost ease. In order to illustrate the concept of EAJs, they are first developed for two-DOF rotations followed by the three-DOF spatial rotations.

3.1 EAJs for Two-DOF Rotations. It is worth noting that for the two-DOF rotations, EAJs consisting of two intersecting revolute joints are equivalent to Euler angles set with two varying and one non-varying angles, and the non-varying angle is always zero, i.e., 0°. For example, YZ EAJs correspond to Euler angle set αYZ or YZα, where α can be the X, Y, or Z axis with 0° rotation about the α axis. Hence, $\mathbf{Q}_\alpha$ is always the identity matrix, and the rotation matrix $\mathbf{Q}_{\alpha YZ}$ or $\mathbf{Q}_{YZ\alpha}$ is nothing but $\mathbf{Q}_{YZ}$. In this section, it will be shown how the EAJs can be developed to represent a two-DOF universal joint using the equivalent rotation matrices obtained in Table 2.

Figure 3 shows a universal joint that connects moving link #M to a reference link #R. Two coordinate frames F_M (O_M-$X_MY_MZ_M$)

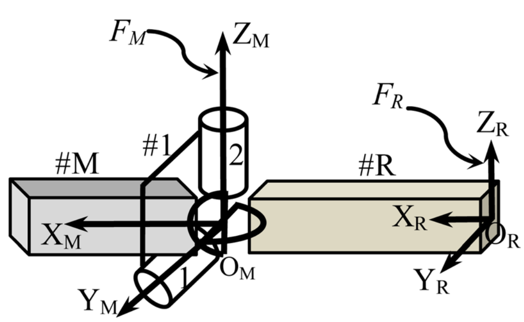


Fig. 3 A universal joint represented by two intersecting revolute joints

Table 4 DH parameters for YZ EAJs

	α_k	θ_k (JV)
1	−90	θ_1
2	90	θ_2

JV: Joint variable.

and F_R (O_R-$X_RY_RZ_R$) are rigidly attached to links #M and #R, respectively. The rotation matrix between these frames can be obtained using any two varying Euler angles. The universal joint can also be described by using two intersecting revolute joints as shown in Fig. 3. Joint 1 connects reference link #R to an imaginary link #1, whereas Joint 2 connects imaginary link #1 with the moving link #M. Now, in order to develop the EAJs representing the rotations equivalent to two varying Euler angles, the DH parameters are extracted from the equivalent rotation matrices obtained for the composite rotations in Sec. 2.2.

If YZ composite rotations are chosen, the DH parameters for the YZ EAJs can be obtained from the equivalent rotation matrix in Eq. (2), i.e., $\mathbf{Q}_{YZ} = \mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}$, where $\mathbf{Q}_{\theta_i}$, for $i=1$, 2, represents the rotation matrix corresponding to the rotations about the Z axis by angle θ_i, whereas $\mathbf{Q}_{\pm X}$ represents the rotation matrix corresponding to the rotation about the X axis by angle $\pm 90^{\circ}$. Thus, $\mathbf{Q}_{YZ}$ is only represented in terms of the rotation matrices representing the rotation about the X and Z axes only. Hence, θ_k, for $k=1$, 2, can be interpreted as the variable DH parameter or the joint angle, whereas the rotation about the X axis, i.e., $\mathbf{Q}_{\pm X}$, may be interpreted as the rotations due to the twist angle α. Moreover, in the definitions of DH parameters the rotation due to the twist angle precedes the one due to the joint angle, as evident from Fig. 9 in the Appendix. Therefore, the DH parameters for the YZ EAJs can be extracted from the expression of the equivalent rotation matrix $\mathbf{Q}_{YZ}$ as follows:

- The first two terms $\mathbf{Q}_{-X}\mathbf{Q}_{\theta_1}$ correspond to the twist angle $\alpha_1 = -90^{\circ}$ and joint angle θ_1.
- The final two terms $\mathbf{Q}_{+X}\mathbf{Q}_{\theta_2}$ represent the twist angle $\alpha_1 = 90^{\circ}$ and the joint angle θ_2.

The DH parameters for the YZ EAJs are listed in Table 4, whereas the DH frames corresponding to the intersecting revolute joints are shown in Fig. 4. The frames are assigned as follows:

- For $\alpha_1 = -90^{\circ}$, axis Z_1 is perpendicular to Z_R, and it represents the joint axis of revolute Joint 1 connecting the imaginary link #1 to the reference link #R.
- For $\alpha_2 = 90^{\circ}$, axis Z_M is the obvious choice for the joint axis of the second revolute joint, as Z_M is perpendicular to Z_1. Joint 2 connects the moving link #M to the imaginary link #1.

The rotation matrix $\mathbf{Q}_{YZ}$ for the YZ EAJs can be derived from the DH parameters of Table 4. It is shown in the fourth row of Table 5. Similarly, other EAJs were obtained for the two-DOF rotations. They are listed in Table 5.

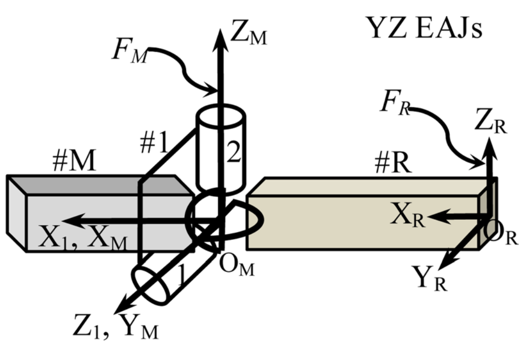


Fig. 4 Representation of DH frames for YZ EAJs

3.2 EAJs for Spatial Rotations. The Euler angles represent rotation in three-dimensional Cartesian space. Hence, they are commonly used to represent a spatial rotation provided by a spherical joint. Figure 5 shows a spherical joint that connects a moving link #M to a reference link #R. The rotation between the frames F_M and F_R can be represented by any Euler angles set. For example, if the ZYZ set is used, one obtains the orientation matrix $\mathbf{Q}_{ZYZ}$ given by Eq. (A2) in the Appendix. For the spherical joint represented with three intersecting joints as shown in Fig. 5, the same rotation matrix can be obtained if the equivalent rotation matrix obtained in Eq. (4) is used to define the DH parameters. The systematic development of the ZYZ and XYZ EAJs will be presented next.

3.2.1 ZYZ-EAJs. The DH parameters for the ZYZ EAJs can be extracted from Eq. (4), i.e., $\mathbf{Q}_{ZYZ} = \mathbf{Q}_{\theta_1}\mathbf{Q}_{-X}\mathbf{Q}_{\theta_2}\mathbf{Q}_{+X}\mathbf{Q}_{\theta_3}$, as follows:

- First, the rotation matrix $\mathbf{Q}_{\theta_1}$ ($\equiv \mathbf{1}\mathbf{Q}_{\theta_1}$, $\mathbf{1}$ being an identity matrix) corresponds to the twist angle $\alpha_1 = 0$ and the joint angle of θ_1.
- The next set of rotation matrices $\mathbf{Q}_{-X}$ and $\mathbf{Q}_{\theta_2}$ correspond to the twist angle $\alpha_2 = -90^{\circ}$ and the joint angle of θ_2.
- Finally, the rotation matrices $\mathbf{Q}_{+X}$ and $\mathbf{Q}_{\theta_3}$ correspond to the twist angle $\alpha_3 = 90^{\circ}$ and the joint angle of θ_3.

The DH parameters thus obtained are listed in Table 6. The corresponding DH frames are shown in Fig. 6. They are assigned as follows:

- For $\alpha_1 = 0^{\circ}$, axis Z_1 is parallel to Z_R. It represents the joint axis of revolute Joint 1 connecting the imaginary link #1 to the reference link #R.
- For $\alpha_2 = -90^{\circ}$, axis Z_2 is orthogonal to Z_1. It represents the joint axis of the revolute Joint 2 connecting the imaginary link #2 to the imaginary link #1. Note that the axis Z_2 is initially parallel to Y_R, as the second Euler angle rotation is about the Y axis.
- For $\alpha_3 = 90^{\circ}$, the third joint axis is orthogonal to Z_2 and parallel to Z_M. It connects the link #M to the imaginary link #2. The axis Z_M is initially parallel to the axis Z_R as the final Euler angle rotation is about the Z axis.

Referring to Fig. 6, as the three revolute joints rotates, the frame F_M attached to #M will change its orientation with respect to the frame F_R attached to #R. Using the DH parameters in Table 6, the rotation matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$, representing the orientation between the intermediate frames are obtained from Eq. (A5) as

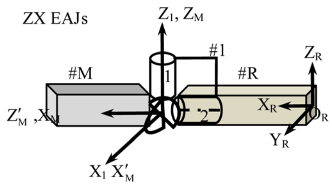
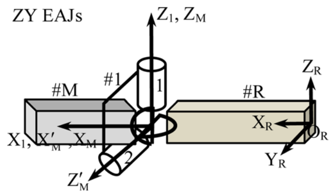
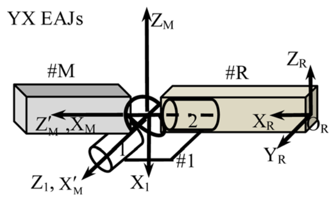
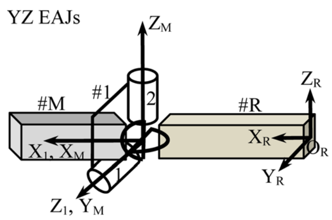
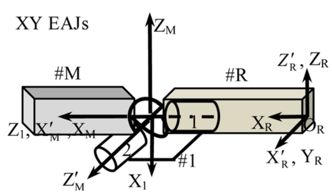
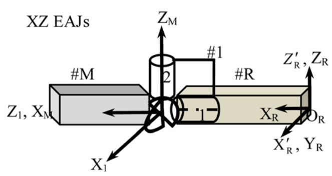
$$\mathbf{Q}_1 \equiv \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_2 \equiv \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ 0 & 0 & 1 \\ -S\theta_2 & -C\theta_2 & 0 \end{bmatrix}, \quad \text{and}$$
$$\mathbf{Q}_3 \equiv \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ 0 & 0 & -1 \\ S\theta_3 & C\theta_3 & 0 \end{bmatrix} \tag{9}$$

It is important to point out here that the matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ above are different from those in Eq. (A1) obtained using the ZYZ Euler angles. The matrix $\mathbf{Q}_{ZYZ}$ ($\equiv \mathbf{Q}_1\,\mathbf{Q}_2\,\mathbf{Q}_3$) providing the overall orientation between #M and #R appears in the second row of Table 8. The elements of $\mathbf{Q}_{ZYZ}$ are nothing but those obtained for the ZYZ Euler angle set in Eq. (A2). This proves that the three intersecting revolute joints shown in Fig. 6 are equivalent to the spherical joint whose rotations are denoted with ZYZ Euler angles. Hence, the revolute joints in Fig. 6 are termed as ZYZ Euler-angle-joints.

3.2.2 XYZ-EAJs. In order to show how the EAJs evolve for asymmetric Euler (or Bryant) angles, the XYZ set is obtained next using Eq. (8) as follows:

- The first term in Eq. (8), $\mathbf{Q}_{+Z}$($\equiv \mathbf{1}\mathbf{Q}_{+Z}$) corresponds to the twist angle $\alpha_0 = 0^{\circ}$ and the joint angle $\theta_0 = 90^{\circ}$.

Table 5 Euler-angle-joints for representation of a universal joint

	Euler angle		DH parameters α_k	DH parameters θ_k	Equivalent EAJs	Rotation matrix using the DH parameters of the EAJs
1	ZX	1 2 3	0 90 −90	θ_1+90 θ_2 −90		$\mathbf{Q}_{ZX} \equiv \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_{-XZ} \equiv \begin{bmatrix} C_1 & -S_1C_2 & S_1S_2 \\ S_1 & C_1C_2 & -C_1S_2 \\ 0 & S_2 & C_2 \end{bmatrix}$
2	ZY	1 2 3	0 −90 90	θ_1 θ_2 0		$\mathbf{Q}_{ZY} \equiv \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_{+X} \equiv \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 \\ S_1C_2 & C_1 & S_1S_2 \\ -S_2 & 0 & C_2 \end{bmatrix}$
3	YX	1 2 3	−90 90 −90	θ_1+90 θ_2+90 −90		$\mathbf{Q}_{YX} \equiv \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_{-XZ} \equiv \begin{bmatrix} C_1 & S_1S_2 & S_1C_2 \\ 0 & C_2 & -S_2 \\ -S_1 & C_1S_2 & C_1C_2 \end{bmatrix}$
4	YZ	1 2	−90 90	θ_1 θ_2		$\mathbf{Q}_{YZ} \equiv \mathbf{Q}_1\mathbf{Q}_2 \equiv \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 \\ S_2 & C_2 & 0 \\ -S_1C_2 & S_1S_2 & C_1 \end{bmatrix}$
5	XY	0 1 2 3	0 90 −90 90	90 θ_1-90 θ_2-90 0		$\mathbf{Q}_{XY} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_{+X} \equiv \begin{bmatrix} C_2 & 0 & S_2 \\ S_1S_2 & C_1 & -S_1C_2 \\ -C_1S_2 & S_1 & C_1C_2 \end{bmatrix}$
6	XZ	0 1 2	0 90 −90	90 θ_1 θ_2-90		$\mathbf{Q}_{XZ} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2 \equiv \begin{bmatrix} C_2 & -S_2 & 0 \\ C_1S_2 & C_1C_2 & -S_1 \\ S_1S_2 & S_1C_2 & C_1 \end{bmatrix}$

Note: (1) $\mathbf{Q}_k$, for $k=1, 2$, represents the rotation matrix associated with the k^{th} set of DH parameters.
(2) $\mathbf{Q}_{+X} = \mathbf{Q}_{X(90)}$, $\mathbf{Q}_{+Z} = \mathbf{Q}_{Z(90)}$, $\mathbf{Q}_{-XZ} = \mathbf{Q}_{X(-90)}\mathbf{Q}_{Z(-90)}$, and $C(\bullet) \equiv \text{Cos}(\bullet)$ and $S(\bullet) \equiv \text{Sin}(\bullet)$.

- The terms $\mathbf{Q}_{+X}, \mathbf{Q}_{\theta_1}$, and $\mathbf{Q}_{-Z}$ correspond to the twist angle $\alpha_1 = 90°$ and the net joint angle of $(\theta_1 - 90°)$, respectively.
- Next, the terms $\mathbf{Q}_{-X}, \mathbf{Q}_{-Z}$, and $\mathbf{Q}_{\theta_2}$ correspond to the twist angle $\alpha_2 = -90°$ and the joint angle of $(\theta_2 - 90)$.
- Finally, the terms $\mathbf{Q}_{+X}$ and $\mathbf{Q}_{\theta_3}$ correspond to the twist angle $\alpha_3 = 90°$ and the net joint angle of θ_3, respectively.

The DH parameters for XYZ EAJs are tabulated in Table 7. The corresponding frame assignments are shown in Fig. 7 and explained below:

- For the constant terms $\alpha_0 = 0°$ and $\theta_0 = 90°$, the DH frame O'_R-$X'_RY'_RZ'_R$ is obtained from frame O_R-$X_RY_RZ_R$ by rotating O_R-$X_RY_RZ_R$ by 90° about Z_R. Both the frames are rigidly attached to the reference link #R.
- For $\alpha_1 = 90°$, axis Z_1 is perpendicular to Z'_R, and it represents the joint axis of revolute Joint 1. Moreover, for the joint angle $(\theta_1 - 90°)$, X_1 is perpendicular to X'_R initially.
- For $\alpha_2 = -90°$, axis Z_2 is perpendicular to Z_1, and it represents the joint axis of revolute Joint 2. The axis Z_2 is parallel to Y_R initially, as the second Euler angle rotation is about the Y axis.
- Finally, for $\alpha_3 = 90°$ and the net joint angle of θ_3, the third joint axis is perpendicular to Z_2. As Z_M is perpendicular to both Z_2 and X_2, it is the obvious choice for the third joint axis.

Using the DH parameters in Table 7 the rotation matrices $\mathbf{Q}_0(=\mathbf{Q}_{+Z})$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ are obtained by using Eq. (A5) as

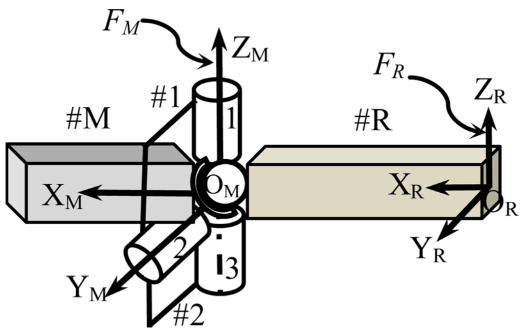


Fig. 5 A spherical joint represented by three intersecting revolute joints

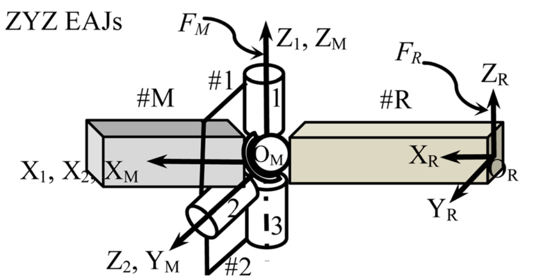


Fig. 6 Representation of DH frames for ZYZ EAJs

Table 6 DH parameters for ZYZ EAJs

	α_k	θ_k (JV)
1	0	θ_1
2	−90	θ_2
3	90	θ_3

JV: Joint variable.

Table 7 DH parameters for XYZ EAJs

	α_k	θ_k (JV)
0	0	90 (Constant)
1	90	θ_1-90
2	−90	θ_2-90
3	90	θ_3

JV: Joint variable.

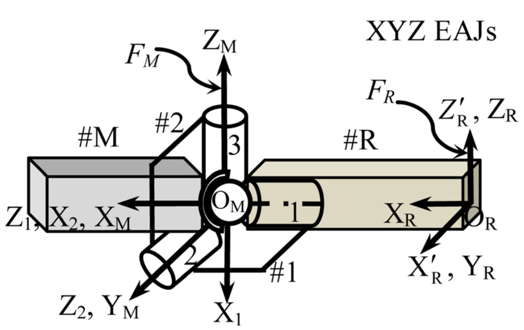


Fig. 7 Representation of DH frames for XYZ EAJs

$$\mathbf{Q}_{+Z} \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_1 \equiv \begin{bmatrix} S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & -1 \\ -C\theta_1 & S\theta_1 & 0 \end{bmatrix},$$
$$\mathbf{Q}_2 \equiv \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 \\ 0 & 0 & 1 \\ -S\theta_2 & -C\theta_2 & 0 \end{bmatrix}, \quad \mathbf{Q}_3 \equiv \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ 0 & 0 & -1 \\ S\theta_3 & C\theta_3 & 0 \end{bmatrix} \tag{10}$$

The overall rotation matrix between the frames F_R and F_M is then obtained as $\mathbf{Q}_{XYZ} = \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$ and given in the eleventh row of Table 8. It is worth mentioning here that while constructing XYZ EAJs an addition constant set of DH parameters, other than three regular sets, is required, as seen in Table 7. This results into one additional constant orientation term $\mathbf{Q}_{+Z}$ at the beginning, as evident form Eq. (10). Moreover, all the EAJs corresponding to the Euler angle sets beginning with a rotation about X, e.g., XYX, XYZ, XZY, require multiplication of constant rotation matrix Q_{+Z} in the beginning. Similarly, the EAJs corresponding to all 12 sets of Euler angles are obtained and shown in Table 8.

It can be summarized that the concept of EAJs not only establishes correlation between the DH parameters and the Euler angles but also facilitates the systematic use of the intersecting revolute joints to describe the Euler angle rotations even though the configuration of the links are defined using the DH parameters. Such correlations were never reported in the literature and, thus, form one of the important contributions of this paper. Moreover, the EAJs also helped in unified representation of multiple-DOF joints.

4 Characteristics of EAJs

In this section some key features of EAJs like the presence of extended DH parameters and differential relations and some issues like non-uniqueness and singularity are addressed.

4.1 Extended DH Parameters. While developing the concept of EAJs, it was found that the definition of some of the EAJs required an additional constant set of DH parameters other than the regular sets corresponding to the DOF of the intersecting revolute joints. These additional sets of the DH parameters are referred to here as extended DH parameters. As a result, a constant rotation matrix is required either in the beginning or at the end as evident form Tables 5 and 8. Interestingly, the existence of extended DH parameters and the corresponding constant rotation matrix depends on the axes about which the Euler angles are defined. This is summarized below.

(1) EAJs having first rotation about the Z or Y axis do not require any constant rotation matrix in the beginning.
(2) If the first rotation is about the X axis, a constant rotation matrix of $\mathbf{Q}_{+Z}$ is required in the beginning.
(3) The EAJs having final rotation about the Z axis do not require any constant rotation at the end.
(4) If the final rotation is about the Y or X axis, it requires a constant rotation matrix of $\mathbf{Q}_{+X}$ or $\mathbf{Q}_{-XZ}$ at the end.

Table 9 shows the extended DH parameters and the resulting requirement of matrix multiplications in the beginning or at the end. From Table 9 it may be seen that the YZ EAJs do not require any extended DH parameters and should be preferred for the representation of a universal joint. Similarly, the symmetric EAJs ZYZ and ZXZ and asymmetric EAJs YXZ are free from the requirement of extended DH parameters. More specifically, only three sets of DH parameters are required to define these EAJs. Hence, one should use ZYZ and ZXZ EAJs to represent spatial rotations if a symmetric set is chosen. On the other hand, YXZ EAJs should be preferred if an asymmetric set is chosen.

4.2 Differential Relations. One of the important applications of the spherical or universal joint representations using the EAJs is the dynamic modeling and simulation of robotic and multibody

Table 8 Euler-angle-joints for representation of spherical joint

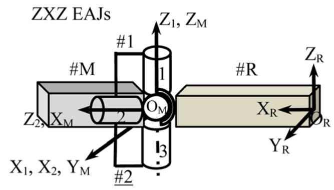
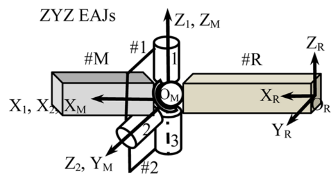
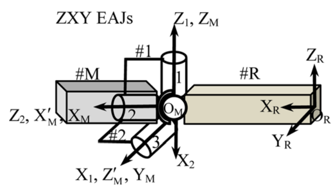
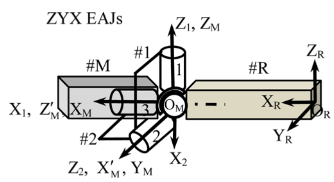
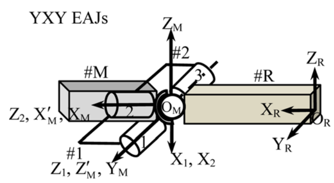
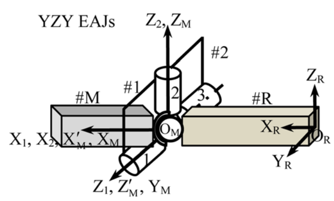
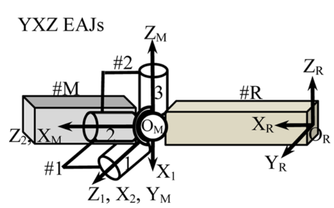
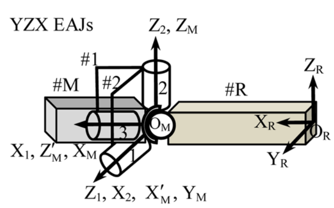
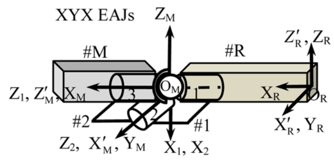
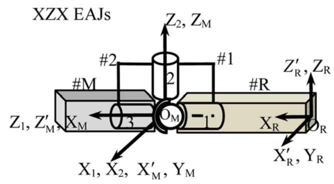
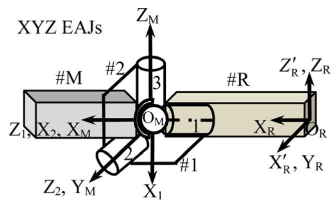
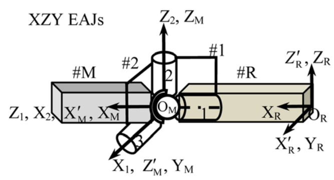
	Euler angle		DH parameters α_k	DH parameters θ_k	Equivalent EAJs	Rotation matrix using the DH parameters of the EAJs
1	ZXZ	1	0	θ_1+90		$\mathbf{Q}_{ZXZ}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\equiv \begin{bmatrix} -S_1C_2S_3+C_1C_3 & -S_1C_2C_3-C_1S_3 & S_1S_2 \\ C_1C_2S_3+S_1C_3 & C_1C_2C_3-S_1S_3 & -C_1S_2 \\ S_2S_3 & S_2C_3 & C_2 \end{bmatrix}$
		2	90	θ_2		
		3	−90	θ_3-90		
2	ZYZ	1	0	θ_1		$\mathbf{Q}_{ZYZ}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\equiv \begin{bmatrix} C_1C_2C_3-S_1S_3 & -C_1C_2S_3-S_1C_3 & C_1S_2 \\ S_1C_2C_3+C_1S_3 & -S_1C_2S_3+C_1C_3 & S_1S_2 \\ -S_2C_3 & S_2S_3 & C_2 \end{bmatrix}$
		2	−90	θ_2		
		3	90	θ_3		
3	ZXY	1	0	θ_1+90		$\mathbf{Q}_{ZXY}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{+X}\equiv \begin{bmatrix} -S_1S_2S_3+C_1C_3 & -S_1C_2 & S_1S_2C_3+C_1S_3 \\ C_1S_2S_3+S_1C_3 & C_1C_2 & -C_1S_2C_3+S_1S_3 \\ -C_2S_3 & S_2 & C_2C_3 \end{bmatrix}$
		2	90	θ_2-90		
		3	−90	θ_3-90		
		4	90	0		
4	ZYX	1	0	θ_1		$\mathbf{Q}_{ZYX}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{-XZ}\equiv \begin{bmatrix} C_1C_2 & C_1S_2S_3-S_1C_3 & C_1S_2C_3+S_1S_3 \\ S_1C_2 & S_1S_2S_3+C_1C_3 & S_1S_2C_3-C_1S_3 \\ -S_2 & C_2S_3 & C_2C_3 \end{bmatrix}$
		2	−90	θ_2+90		
		3	90	θ_3+90		
		4	−90	−90		
5	YXY	1	−90	θ_1+90		$\mathbf{Q}_{YXY}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{+X}\equiv \begin{bmatrix} -S_1C_2S_3+C_1C_3 & S_1S_2 & S_1C_2C_3+C_1S_3 \\ S_2S_3 & C_2 & -S_2C_3 \\ -C_1C_2S_3-S_1C_3 & C_1S_2 & C_1C_2C_3-S_1S_3 \end{bmatrix}$
		2	90	θ_2		
		3	−90	θ_3-90		
		4	90	0		
6	YZY	1	−90	θ_1		$\mathbf{Q}_{YZY}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{+X}\equiv \begin{bmatrix} C_1C_2C_3-S_1S_3 & -C_1S_2 & C_1C_2S_3+S_1C_3 \\ S_2C_3 & C_2 & S_2S_3 \\ -S_1C_2C_3-C_1S_3 & S_1S_2 & -S_1C_2S_3+C_1C_3 \end{bmatrix}$
		2	90	θ_2		
		3	−90	θ_3		
		4	90	0		
7	YXZ	1	−90	θ_1+90		$\mathbf{Q}_{YXZ}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3 \begin{bmatrix} S_1S_2S_3+C_1C_3 & S_1S_2C_3-C_1S_3 & S_1C_2 \\ C_2S_3 & C_2C_3 & -S_2 \\ C_1S_2S_3-S_1C_3 & C_1S_2C_3+S_1S_3 & C_1C_2 \end{bmatrix}$
		2	90	θ_2+90		
		3	−90	θ_3-90		
8	YZX	1	−90	θ_1		$\mathbf{Q}_{YZX}\equiv\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{-XZ}\equiv \begin{bmatrix} C_1C_2 & -C_1S_2C_3+S_1S_3 & C_1S_2S_3+S_1C_3 \\ S_2 & C_2C_3 & -C_2S_3 \\ -S_1C_2 & S_1S_2C_3+C_1S_3 & -S_1S_2S_3+C_1C_3 \end{bmatrix}$
		2	90	θ_2+90		
		3	90	θ_3		
		4	−90	−90		

Table 8. *Continued*

	Euler angle	DH parameters			Equivalent EAJs	Rotation matrix using the DH parameters of the EAJs
			α_k	θ_k		
9	XYX	0	0	90		$\mathbf{Q}_{XYX} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{-XZ} \equiv \begin{bmatrix} C_2 & S_2S_3 & S_2C_3 \\ S_1S_2 & -S_1C_2S_3+C_1C_3 & -S_1C_2C_3-C_1S_3 \\ -C_1S_2 & C_1C_2S_3+S_1C_3 & C_1C_2C_3-S_1S_3 \end{bmatrix}$
		1	90	θ_1-90		
		2	−90	θ_2		
		3	90	θ_3+90		
		4	−90	−90		
10	XZX	0	0	90		$\mathbf{Q}_{XZX} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{-XZ} \equiv \begin{bmatrix} C_2 & -S_2C_3 & S_2S_3 \\ C_1S_2 & C_1C_2C_3-S_1S_3 & -C_1C_2S_3-S_1C_3 \\ S_1S_2 & S_1C_2C_3+C_1S_3 & -S_1C_2S_3+C_1C_3 \end{bmatrix}$
		1	90	θ_1		
		2	−90	θ_2		
		3	90	θ_3		
		4	−90	−90		
11	XYZ	0	0	90		$\mathbf{Q}_{XYZ} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3 \equiv \begin{bmatrix} C_2C_3 & -C_2S_3 & S_2 \\ S_1S_2C_3+C_1S_3 & -S_1S_2S_3+C_1C_3 & -S_1C_2 \\ -C_1S_2C_3+S_1S_3 & C_1S_2S_3+S_1C_3 & C_1C_2 \end{bmatrix}$
		1	90	θ_1-90		
		2	−90	θ_2-90		
		3	90	θ_3		
12	XZY	0	0	90		$\mathbf{Q}_{XZY} \equiv \mathbf{Q}_{+Z}\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_{+X} \equiv \begin{bmatrix} C_2C_3 & -S_2 & C_2S_3 \\ C_1S_2C_3+S_1S_3 & C_1C_2 & C_1S_2S_3-S_1C_3 \\ S_1S_2C_3-C_1S_3 & S_1C_2 & S_1S_2S_3+C_1C_3 \end{bmatrix}$
		1	90	θ_1		
		2	−90	θ_2-90		
		3	−90	θ_3		
		4	90	0		

Note: (1) $\mathbf{Q}_k$, for $k = 1, 2, 3$, represents the rotation matrix associated with the k^{th} set of DH parameters.
(2) $\mathbf{Q}_{+X} = \mathbf{Q}_{X(90)}$, $\mathbf{Q}_{+Z} = \mathbf{Q}_{Z(90)}$, $\mathbf{Q}_{-XZ} = \mathbf{Q}_{X(-90)}\mathbf{Q}_{Z(-90)}$, and $C(\bullet) \equiv \text{Cos}(\bullet)$ and $S(\bullet) \equiv \text{Sin}(\bullet)$.

systems. In this section, the differential relationship, mainly, the relationship between angular velocities of the moving and reference links in terms of the joint rates of the EAJs is obtained. Referring to Fig. 6, angular velocity ($\boldsymbol{\omega}_1$) of the link #1 in terms of the reference link #R is written as

$$\boldsymbol{\omega}_1 = \mathbf{e}_1\dot{\theta}_1 \tag{11}$$

where $\mathbf{e}_1$ is the unit vector along the axis of rotation of the first joint, and $\dot{\theta}_1$ is the rate of the Euler angle. Similarly the angular velocities for the second and moving links, i.e., #2 and #M, with respect to #R are given by

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \mathbf{e}_2\dot{\theta}_2 = \mathbf{e}_1\dot{\theta}_1 + \mathbf{e}_2\dot{\theta}_2 \tag{12}$$

$$\boldsymbol{\omega}_M = \boldsymbol{\omega}_2 + \mathbf{e}_3\dot{\theta}_3 = \mathbf{e}_1\dot{\theta}_1 + \mathbf{e}_2\dot{\theta}_2 + \mathbf{e}_3\dot{\theta}_3 \tag{13}$$

Equation (13) represents the angular velocity of the moving link #M with respect to the reference link #R connected by EAJs and can be represented in a compact form as

$$\boldsymbol{\omega}_M = \mathbf{P}\dot{\boldsymbol{\theta}}, \text{ where } \mathbf{P} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \text{ and } \dot{\boldsymbol{\theta}} = [\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3]^T \tag{14}$$

where $\mathbf{P}$ is the 3×3 transformation matrix and $\dot{\boldsymbol{\theta}}$ is the three-dimentional vector of rate of EAJs. The expression of $\mathbf{P}$ will vary depending on the set of EAJs.

In this subsection expression of $\mathbf{P}$ is derived for the ZYZ EAJs. The unit vector $\mathbf{e}_i$, for $i = 1, 2, 3$, in the i^{th} frame, is given as $[\mathbf{e}_i]_i = [0 \ 0 \ 1]^T$. Accordingly, $\mathbf{e}_i$ in the moving frame F_M, i.e., $[\mathbf{e}_i]_M$, can be obtained as

$$[\mathbf{e}_3]_M = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{e}_2]_M = \mathbf{Q}_3^T[\mathbf{e}_2]_2 = \begin{bmatrix} S\theta_3 \\ C\theta_3 \\ 0 \end{bmatrix}, \quad \text{and}$$
$$[\mathbf{e}_1]_M = \mathbf{Q}_3^T\mathbf{Q}_2^T[\mathbf{e}_1]_1 = \begin{bmatrix} S\theta_2C\theta_3 \\ -S\theta_2S\theta_3 \\ C\theta_2 \end{bmatrix} \tag{15}$$

Table 9 Required extended DH parameters/constant matrix multiplication for the EAJs

	Extended DH parameters/constant matrix multiplication			
	Not required	Required at the beginning	Required at the end	Required at both the ends
EAJs with two rotations	YZ	XZ	YX, ZX, ZY	XY
EAJs with three rotations	ZYZ, ZXZ, YXZ	XYZ	ZYX, YZX, ZXY, YXY, YZY	XYX, XZX, XZY

where $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ were obtained in Eq. (9). Using Eqs. (14) and (15), the expression for matrix $\mathbf{P}$ is written as

$$\mathbf{P} = \begin{bmatrix} S\theta_2 C\theta_3 & S\theta_3 & 0 \\ -S\theta_2 S\theta_3 & C\theta_3 & 0 \\ C\theta_2 & 0 & 1 \end{bmatrix} \tag{16}$$

Interestingly, the expression in Eq. (16) is nothing but the one shown in Eq. (A4) for the ZYZ Euler angles. It is worth noting that for any two moving links in a multibody system, Eq. (14) should be interpreted between the link #k and its parent link #($k-1$) instead of #M and #R, respectively.

4.3 Non-Uniqueness. Non-uniqueness of a rotation representation using Euler angles is a well-known issue for a given rotation matrix. It is not different with EAJs, as both lead to the same rotation matrix. For a given rotation matrix $\mathbf{Q}$, i.e.,

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \tag{17}$$

the angles of the EAJs may be found in a manner similar to those of Euler angles by comparing the elements of $\mathbf{Q}$ in Eq. (17) with those of $\mathbf{Q}_{ZYZ}$ in the second row of Table 8 as

$$C\theta_2 = q_{33}; S\theta_2 = \pm\sqrt{1 - q_{33}^2} \tag{18}$$

$$C\theta_1 = \frac{q_{13}}{S\theta_2};\ S\theta_1 = \frac{q_{23}}{S\theta_2} \tag{19}$$

$$C\theta_3 = \frac{-q_{31}}{S\theta_2};\ S\theta_3 = \frac{q_{32}}{S\theta_2} \tag{20}$$

It is clear from Eq. (18), that there are two solutions for θ_2, whereas for each value of θ_2, Eqs. (19) and (20) provide one solution. Hence, the problem of calculating angles of EAJs from the given rotation matrix is non-unique.

4.4 Singularity. Singularity is an inherent problem with representation of spatial rotations with three parameters [1] including the EAJs. This is evident form Eqs. (19) and (20), where the angles θ_1 and θ_3 cannot be solved if $S\theta_2 = 0$. The same can be checked form Eq. (16), where determinant of the transformation matrix $\mathbf{P}$ vanishes. As a consequence, one cannot obtain $\dot{\boldsymbol{\theta}}$ from Eq. (14) for a given $\boldsymbol{\omega}$. This is also referred to as a phenomenon called "Gimbal Lock" [7]. The condition occurs when the axis of Joint 1 coincides with that of Joint 3. All the symmetric EAJs suffer from singularity for $\theta_2 = 0$ or $n\pi$, whereas all the asymmetric EAJs are singular for $\theta_2 = (2n-1)\pi/2$. In reality, most of the spherical joints have restricted motion, and areas of Gimbal Lock can be kept outside the domain of movement of the joint. Detailed discussion on the singularity of EAJs is beyond the scope of this paper. However, one may use the singularity avoidance algorithm developed for Euler angles [8,9] to avoid any singularity in EAJs.

5 Conclusions

This paper presents a novel concept of Euler-angle-joints by introducing the DH parameterization of the well-known Euler angles used to describe three-dimensional spatial rotation. Euler-angle-joints are nothing but orthogonally intersecting revolute joints whose axes are defined using the well-known DH parameters. They are so connected by imaginary links of zero lengths and masses that they represent a particular set of Euler angles. Hence, the proposed concept not only establishes a correlation between the DH parameters and the Euler angles but also facilitates the systematic use of the intersecting revolute joints to describe the Euler angle rotations even though the configurations of links are defined using the DH parameters. Such correlations were never reported in the literature and, thus, they form an important contribution of this paper. While developing the EAJs, evolution of the DH parameters for different rotation sequences of Euler angles have been investigated. They are summarized in Table 9.

The concept of EAJs lends its utility in the unified representation of one-, two-, and three-DOF joints, i.e., revolute, universal, and spherical, respectively. Such unification makes an algorithm development for kinematic and dynamic analysis much simpler, which was not possible with the original definition of the Euler angles.

Appendix A

This section presents definitions of Euler angles and DH parameters using the notations used in this paper.

A.1 Euler Angles

According to Euler's rotation theorem [1], any three-dimensional spatial rotation can be described using three sequential angles of rotations about three independent axes. These angles of rotation are called Euler angles. Figures 8(*a*)–8(*c*) show the sequence of rotation, given by (a) an angle θ_1 about the Z_M axis, (b) an angle θ_2 about the rotated Y_M axis, and (c) an angle θ_3 about the current Z_M axis, respectively. The frame F_R (O_R-$X_RY_RZ_R$) and F_M (O_M-$X_MY_MZ_M$) denote the reference frame and moving frame, respectively. The three angles θ_1, θ_2, and θ_3 are called the ZYZ Euler angles as per the sequential rotations performed about Z_M, new Y_M, and new Z_M. In a similar way, one can define all twelve such sets of Euler angles [10].

If the elementary rotations about the Z_M, new Y_M, and new Z_M axes are θ_1, θ_2, and θ_3, then the respective rotation matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ are given by

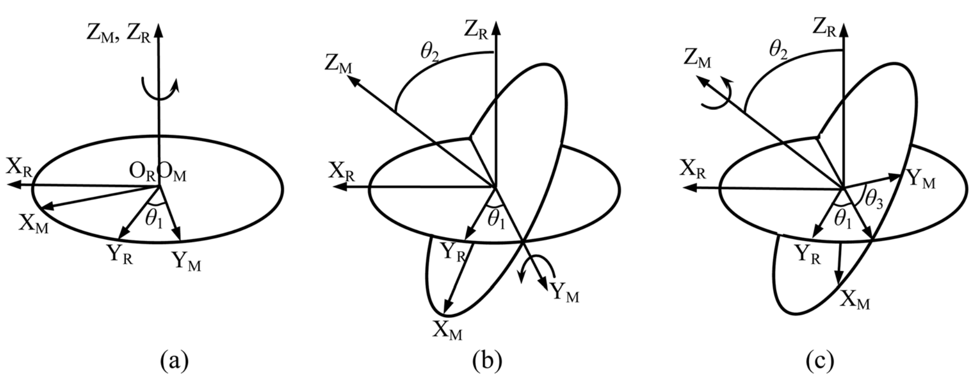


Fig. 8 ZYZ Euler angles

$$\mathbf{Q}_1 \equiv \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 \\ S\theta_1 & C\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_2 \equiv \begin{bmatrix} C\theta_2 & 0 & S\theta_2 \\ 0 & 1 & 0 \\ -S\theta_2 & 0 & C\theta_2 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{Q}_3 \equiv \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ S\theta_3 & C\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A1}$$

The overall rotation matrix $\mathbf{Q}$ between frames F_M and F_R is then obtained as

$$\mathbf{Q} = \mathbf{Q_1Q_2Q_3} = \begin{bmatrix} C\theta_1C\theta_2C\theta_3 - S\theta_1S\theta_3 & -C\theta_1C\theta_2S\theta_3 - S\theta_1C\theta_3 & C\theta_1S\theta_2 \\ S\theta_1C\theta_2C\theta_3 + C\theta_1S\theta_3 & -S\theta_1C\theta_2S\theta_3 + C\theta_1C\theta_3 & S\theta_1S\theta_2 \\ -S\theta_2C\theta_3 & S\theta_2S\theta_3 & C\theta_2 \end{bmatrix} \tag{A2}$$

The differential relationship between the Euler angles, i.e., the relation between rates of Euler angles and angular velocity of frame F_M with respect to frame F_R, can be obtained using the property $\tilde{\boldsymbol{\omega}} \equiv \mathbf{Q}^T\dot{\mathbf{Q}}$ [1], where $\tilde{\boldsymbol{\omega}}$ is the skew-symmetric matrix associated with the three-dimnesionl vector of angular velocity, as

$$\boldsymbol{\omega} = \boldsymbol{P}\dot{\boldsymbol{\theta}} \tag{A3}$$

where the 3×3 transformation matrix $\mathbf{P}$ and the three-dimensional vector $\dot{\boldsymbol{\theta}}$ are given by

$$\mathbf{P} = \begin{bmatrix} S\theta_2C\theta_3 & S\theta_3 & 0 \\ -S\theta_2S\theta_3 & C\theta_3 & 0 \\ C\theta_2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_2 \end{bmatrix} \tag{A4}$$

The columns of the matrix $\mathbf{P}$ represent unit vectors along the Z, Y, and Z axes in the moving frame F_M.

A.2 Denavit-Hartenberg Parameters

In this section, the Denavit-Hartenberg notation, as adopted from Khalil and Kleinfinger [11], is presented for completeness. For that, a coordinate frame is attached to each link. The frame O_k-$X_kY_kZ_k$, denoted by F_k, is rigidly attached to link #k, as shown in Fig. 9. The joint k couples the links #($k-1$) and #k. The axis Z_k represents the k^{th} joint axis. Moreover, the origin of F_k, namely, O_k is located at a point where the common normal to Z_k and Z_{k+1} intersects Z_k, whereas the common normal defines the axis X_k. Furthermore, the axis Y_k is such that the axes X_k, Y_k, and Z_k form a right-handed triad. The coordinate frame is referred to as a DH frame. Note that the frame attached to the fixed link, i.e., O_0-$X_0Y_0Z_0$, can be chosen arbitrarily and hence one can choose Z_0 coincident with Z_1. Once the link frames are established using the above scheme, the position and the orientation between any two frames, say, F_{k-1} and F_k, can be specified using four parameters known as DH parameters. These parameters are defined as

- twist angle (α_k), the angle between Z_{k-1} and Z_k about X_{k-1}
- link length (a_k), the distance from Z_{k-1} to Z_k along X_{k-1}
- joint offset (b_k), the distance from X_{k-1} to X_k along Z_k
- joint angle (θ_k), the angle between X_{k-1} and X_k about Z_k

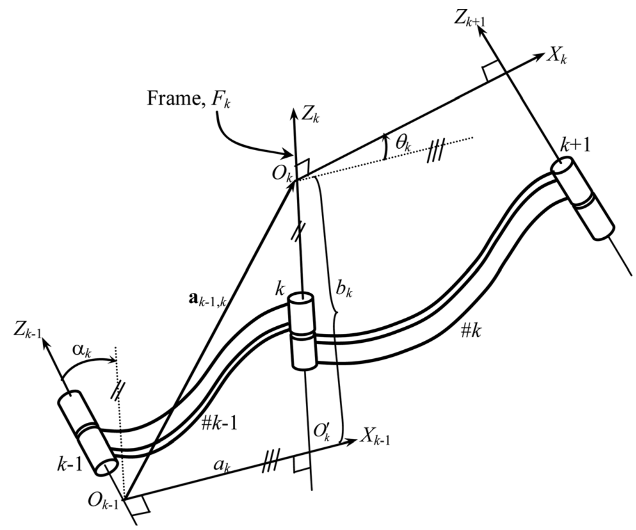


Fig. 9 The Denavit and Hartenberg (DH) parameters and frames

Depending on the type of joints, i.e., revolute or prismatic, θ_k or b_k is a variable quantity while the other parameters are constant. Based on the definition of the above DH parameters, the 3×3 orientation matrix $\mathbf{Q}_k$ defining the orientation of F_k with respect to F_{k-1} can be expressed [11] as

$$\mathbf{Q}_k \equiv \mathbf{Q}_{X(\alpha_k)}\mathbf{Q}_{Z(\theta_k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha_k & -S\alpha_k \\ 0 & S\alpha_k & C\alpha_k \end{bmatrix} \begin{bmatrix} C\theta_k & -S\theta_k & 0 \\ S\theta_k & C\theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta_k & -S\theta_k & 0 \\ S\theta_kC\alpha_k & C\theta_kC\alpha_k & -S\alpha_k \\ S\theta_kS\alpha_k & C\theta_kS\alpha_k & C\alpha_k \end{bmatrix} \tag{A5}$$

References

[1] Shabana, A. A., 2001, *Computational Dynamics*, Wiley, New York.
[2] Shuster, M. D., 1993, "A Survey of Attitude Representations," J. Astronaut. Sci., **41**(4), pp. 439–517.
[3] Nikravesh, P. E., 1988, *Computer-Aided Analysis of Mechanical Systems*, Prentice-Hall, Englewood Cliffs, New Jersey.
[4] Duffy, J., 1978, "Displacement Analysis of the Generalized RSSR Mechanism," Mech. Mach. Theory, **13**, pp. 533–541.
[5] Chaudhary, H., and Saha, S. K., 2007, "Constraint Wrench Formulation for Closed-Loop Systems Using Two-Level Recursions," ASME J. Mech. Des., **129**, pp. 1234–1242.
[6] Denavit, J., and Hartenberg, R. S., 1955, "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," J. Appl. Mech., **22**, pp. 215–221.
[7] Wittenburg, J., 2008, *Dynamics of Multibody systems*, Springer, Berlin.
[8] Shuster, M. D., and Oh, S. D., 1981, "Three-Axis Attitude Determination From Vector Observation," J. Guid. Control Dyn., **4**(1), pp. 70–77.
[9] Singla, P., Mortari, D., and Junkins, J. L., 2004, "How to Avoid Singularity for Euler Angle Set?," Proceedings of the AAS Space Flight Mechanics Conference, Hawaii.
[10] Craig, J. J., 2006, *Introduction to Robotics, Mechanics and Control*, Pearson Education, Delhi.
[11] Khalil, W., Kleinfinger, J., 1986, "A New Geometric Notation for Open and Closed-Loop Robots," IEEE Int. Conf. Rob. Autom., **3**, pp. 1174–1179.